Algebraic Topology

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Introduction

Algebraic topology is a large and complicated array of tools that provide a framework for measuring geometric and algebraic objects with numerical and algebraic invariants. The original motivation was to help distinguish and eventually classify topological spaces up to homeomorphism or up to a weaker equivalence called homotopy type. But the subject has turned out to have a vastly wider range of applicability. In the cases of the most interest, the objects to which we apply the invariants of algebraic topology arise naturally in geometric, analytic, and algebraic studies. The power of algebraic topology is the generality of its application. The tools apply in situations so disparate as seemingly to have nothing to do with each other, yet the common thread linking them is algebraic topology. One of the most impressive arguments by analogy of twentieth century mathematics is the work of the French school of algebraic geometry, mainly Weil, Serre, Grothendieck, and Deligne, to apply the machinery of algebraic topology to projective varieties defined over finite fields in order to prove the Weil Conjectures. On the face of it these conjectures, which dealt with counting the number of solutions over finite fields of polynomial equations, have nothing to do with usual topological spaces and algebraic topology. The powerful insight of the French school was to recognize that in fact there was a relationship and then to establish the vast array of technical results in algebraic geometry over finite fields necessary to implement this relationship.

Let me list some of the contexts where algebraic topology is an integral part. It is related by deRham's theorem to differential forms on a manifold, by Poincaré duality to the study of intersection of cycles on manifolds, and by the Hodge theorem to periods of holomorphic differentials on complex algebraic manifolds. Algebraic topology is used to compute the infinitessimal version of the space of deformations of a complex analytic manifold (and in particular, the dimension of this space). Similarly, it is used to compute the infinitessimal space of deformations of a linear representation of a finitely presented group. In another context, it is used to compute the space of sections of a holomorphic vector bundle. In a more classical vein, it is used to compute the number of handles on a Riemann surface, estimate the number of critical points of a real-valued function on a manifold, estimate the number of fixed points of a self-mapping of a manifold, and to measure how much a vector bundle is twisted. In more algebraic contexts, algebraic topology allows one to understand short exact sequences of groups and modules over a ring, and more generally longer extensions. Lastly, algebraic topology can be used to define the cohomology groups of groups and Lie algebras, providing important invariants of these algebraic objects.

A quote from Lefschetz seems appropriate to capture the spirit of the subject: after a long and complicated study of pencils of hypersurfaces on algebraic varieties he said, "we have succeeded in planting the harpoon of algebraic topology in the whale of algebraic geometry." One part of this image needs amplifying – namely, if one views algebraic topology as a harpoon, then one must see it as a harpoon with a complicated internal structure; inside there are many hidden working parts. The main trouble with algebraic topology is that there are many different approaches to defining the basic objects — homology and cohomology groups. Each approach brings with it a fair amount of required technical baggage, be it singular chains and the algebra of chain complexes in one approach, or derived functors in another. Thus, one must pay a fairly high price and be willing to postpone the joys of the beautiful applications for quite awhile as one sloggs through the basic constructions and proves the basic results. Furthermore, possibly the most striking feature of the subject, the interrelatedness (and often equality) of the theories resulting from different approaches requires even more machinery, for at its heart it is saying that two completely different constructions yield related (or the same answers).

My aim in this course is to introduce you to several approaches to homology and cohomology and indicate results in various geometric and algebraic contexts that flow from judicious uses of homology and cohomology. The level of background that is assumed will vary greatly: when we are considering applications, I will assume whatever is necessary from the area to which we are applying algebraic topology in order to establish the results.

1 Homology

We begin with three different constructions which will generalize to three different, but closely related homology (and cohomology) theories.

1.1 The Simplest Homological Invariants

In this introduction to homology, we begin with some very simple examples of algebraic invariants. These are immediately defined and easy to compute. One may wonder why we are drawing attention to them, since they may seem somewhat forced. The reason for the attention is that, as we shall see after we define homology in all its glory, these are the lowest dimensional homological invariants. Thus, one can view homology as a vast higher dimensional generalization of the fairly obvious invariants we introduce here.

1.1.1 Zeroth Singular Homology

Let X be a topological space and consider the free abelian group $S_0(X)$ with basis the points of X. That is to say, an element of this group is a finite integral linear combination of the form

$$\sum_{\{p\in X\}} n_p[p].$$

This is a finite sum in the sense that all but finitely many of the integral coefficients n_p are zero. Distinct sums represent different elements of the group and one adds sums in the obvious way. At this point the topology of X plays no role – only the points of X are important. The zero element in this group is the empty sum. Of course, the elements $[p]$ as p ranges over the points of X forms a basis for this free abelian group. We call this group the *group of singular zero chains on X*.

The topology of X plays a role when we introduce an equivalence relation on $S_0(X)$. We form a quotient of $S_0(X)$ by setting $[p] \equiv [q]$ if there is a path beginning at p and ending at q, i.e., a continuous map from the unit interval $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. It is easy to see that this is indeed an equivalence relation, and that it induces an equivalence relation on $S_0(X)$. The quotient group of this equivalence relation, $S_0(X)/\sim$ is denoted $H_0(X)$ and is called *the zeroth singular homology group* of X. It is easy to compute.

Lemma 1.1.1. *$H_0(X)$ is the free abelian group generated by the set of path components of X. In particular, if X is path connected, then $H_0(X)$ is isomorphic to $\mathbb{Z}$.*

Proof. Let $\pi_0(X)$ denote the set of path components of X. We have a homomorphism $S_0(X) \to \oplus_{A \in \pi_0(X)} \mathbb{Z}$ defined by sending $\sum_{p \in X} n_p[p]$ to $\sum_{p \in X} n_p[A(p)]$ where $A(p) \in \pi_0(X)$ is the path component containing p. This homomorphism is clearly compatible with the equivalence relation, and hence defines a homomorphism $H_0(X) \to \oplus_{A \in \pi_0(X)} \mathbb{Z}$. It is onto since $[p]$ maps to the element which is one times the component containing p, and these form a basis for the range.

For each path component A choose a point $a \in A$ and for each $p \in A$ choose a path γ_p from a to p. Then $\sum_{p \in A} n_p[p]$ is equivalent to $\sum_{p \in A} n_p a$. Suppose $\alpha = \sum_p n_p[p]$ maps to zero. This means that for each $A \in \pi_0(X)$ we have $\sum_{p \in A} n_p = 0$, and in light of the above equivalences implies that the α is zero in $H_0(X)$. □

We could also define $H_0(X;\mathbb{Q})$ and $H_0(X;\mathbb{R})$ by replacing the $\mathbb{Z}$ coefficients by rational or real coefficients in the above constructions. The resulting groups are then vector spaces over $\mathbb{Q}$ or $\mathbb{R}$.

The homology $H_0(X)$ is a functor from the category of topological spaces and continuous maps to the category of abelian groups. That is to say, if $f : X \to Y$ is a continuous map, then there is an induced mapping $H_0(X) \to H_0(Y)$, and this operation respects compositions and sends the identity map of X to itself to the identity homomorphism on $H_0(X)$. For more on categories and functors see appendix A.

1.1.2 Zeroth deRham Cohomology

In this section M is a smooth ($= C^\infty$) manifold. We define $\Omega^0(M)$ to be the $\mathbb{R}$-vector space of smooth functions on M. These are the *deRham zero cochains*. We define $H^0_{\mathrm{dR}}(M)$, the *zeroth deRam cohomology group*, to be the subgroup of $\Omega^0(M)$ consisting of functions f for which $df = 0$. These of course are the locally constant functions on M, and hence are functions constant on each component of M.

Lemma 1.1.2. *$H^0_{\mathrm{dR}}(M)$ is identified with the $\mathbb{R}$-vector space of functions from the set of components, or equivalently the set of path components, of M to $\mathbb{R}$. In particular, if M is connected, $H^0_{dR}(M)$ is isomorphic as a real vector space to $\mathbb{R}$.*

The deRham cohomology is a contravariant functor from the category of smooth manifolds and smooth maps to the category of real vector spaces. That is to say, if $f : M \to N$ is a smooth map of smooth manifolds, then there is an induced map $f^* : H^0_{\mathrm{dR}}(N) \to H^0_{dR}(M)$. This association preserves compositions and sends identities to identities.

There are three differences between the construction of the zeroth deRham cohomolgy group and the construction of the zeroth singular homology group. First, the deRham construction applies only to smooth manifolds, not all topological spaces. Secondly, the construction takes place in the category of real vector spaces instead of abelian groups. These two changes are summarized by saying that the zeroth deRham cohomology is a functor from the category of smooth manifolds and smooth maps to the category of real vector spaces. Thirdly, the deRham cohomology group is a subgroup of the zero cochains, whereas before the homology group is a quotient of the zero chains. This duality is indicated by the change in terminology from homology to cohomology. It is also reflected in the fact that deRham cohomology is a contravariant functor.

Notice that there is a natural pairing

$$H^0_{\mathrm{dR}}(M) \otimes H_0(M) \to \mathbb{R}$$

given by

$$f \otimes \sum n_p[p] \mapsto \sum n_p f(p).$$

One sees easily that if $f \in H^0_{\mathrm{dR}}(M)$ then the evaluation of f on $S_0(X)$ is constant on equivalence classes and hence passes to a well-defined map on the quotient. If we replace $H_0(X)$ by $H_0(X; \mathbb{R})$ then the pairing becomes a perfect pairing identifying $H^0_{\mathrm{dR}}(X)$ with the dual $\mathbb{R}$-vector space to $H_0(X; \mathbb{R})$.

1.1.3 Zeroth Čech Cohomology

Let X be a topological space, and let $\{U_\alpha\}_{\{\alpha \in A\}}$ be an open covering of X. We define the *Čech zero cochains* with respect to this open covering, $\check{C}^0(X; \{U_\alpha\})$, to be the group of all $\{f_\alpha\}_{\{\alpha \in A\}}$, where f_α is a locally constant function from U_α to $\mathbb{Z}$. The *zeroth Čech cohomology group* $\check{H}^0(X; \{U_\alpha\})$ is the subgroup of all $\{f_\alpha\}$ for which for every $\alpha, \beta \in A$ we have $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$. Clearly, these form an abelian group under addition. Elements of this group are called *cocycles*.

Lemma 1.1.3. *For any open covering of X, the group $H^0(X; \{U_\alpha\})$ is isomorphic to the group of locally constant functions from X to $\mathbb{Z}$. This is the free abelian group of functions from the set of components of X to $\mathbb{Z}$.*

Proof. Let $\{U_a\}$ be a covering of X by open sets and let $\{f_a\}$ be a Čech zero cochain with respect to this covering. Since for all pairs of indices a, b we have $f_a\big|(U_a \cap U_b) = f_b\big|(U_a \cap U_b)$, the local functions f_a glue together to define a function f on X. That is to say, there is a unique function $f : X \to \mathbb{Z}$ with the property that for each index a we have $f|U_a = f_a$.

Since the f_a are locally constant and the U_a are open subsets, it follows that f is locally constant. This defines a homomorphism from $\check{H}^0(X;\{U_a\})$ to the group of locally constant functions from X to $\mathbb{Z}$. Conversely, given a locally constant function f on X, for each index a we let f_a be the restriction $f|U_a$. This, of course, is a locally constant function on U_a and $f_a|U_a \cap U_b = f_b|U_a \cap U_b$. Thus this defines a homomorphism from the group of locally constant functions on X to $\check{H}^0(X;\{U_a\})$. It is clear that these constructions produce inverse homomorphisms. □

To define the zeroth Čech cohomology of X, independent of any open covering, we consider all open coverings $\{U_\alpha\}_\alpha$. The collection of open covers is ordered by refinement: A covering $\{V_\beta\}_{\beta\in B}$ is smaller than a covering $\{U_\alpha\}_{\alpha\in A}$ if for each $\beta \in B$ there is an $\alpha(\beta) \in A$ with the property that $V_\beta \subset U_{\alpha(\beta)}$. If we choose a refinement map, that is to say a map $\rho : B \to A$ with the property that $V_\beta \subset U_{\rho(\beta)}$ for all β, then we can define a map ρ_* from the zero Čech cochains $\check{C}^0(X;\{U_\alpha\})$ to $\check{C}^0(X;\{V_\beta\})$. It is defined as follows. Let $\{f_\alpha\} \in \check{C}^0(X;\{U_\alpha\})$. For each β, set $\rho_*(\{f_\alpha\}) = \{f_{\rho(\beta)}|_{V_\beta}\}$. The map ρ_* on the level of cochains depends on the choice of refinement function ρ, but we have:

Lemma 1.1.4. *If $\{f_\alpha\} \in \check{C}^0(X;\{U_\alpha\})$ is a cocycle, then $\rho_*\{f_\alpha\} \in \check{C}^0(X;\{V_\beta\})$ is a cocyle which is independent of the choice of refinement mapping ρ.*

Proof. $\{f_\alpha\}$ is a cocycle if and only if there is a locally constant function f on X such that $f_\alpha = f|_{U_\alpha}$. Then for any refinement mapping ρ, $\rho_*\{f_\alpha\}$ is simply the cocyle given by restricting f to the V_β. □

The zeroth Čech cohomology is defined as the direct limit of $\check{H}^0(X;\{U_\alpha\})$ and the maps induced by refinements, as $\{U_\alpha\}$ ranges over all coverings. From the above proof it follows immediately that this group is identified with the group of locally constant integral valued functions on X. Set up this way the zeroth Čech cohomology is a functor from the category of topological spaces and continuous maps to the category of abelian groups.

Remark 1.1.5.
1. Again we call this group a cohomology group, and notice that it is a subgroup of the cochain group.
2. The fact that one can glue together local functions defined on open sets to give a global function if and only if the local functions agree on the overlaps can be extended to other contexts. This property is encoded in the notion of a sheaf on a topological space. The first sheaves one encounters are sheaves of functions (locally constant, continuous, smooth, arbitrary). But there are many other kinds of sheaves which play extremely important roles in algebraic geometry, commutative algebra, and complex analytic geometry.

Exercise 1.1.6. *Show that there is a natural pairing*

$$\check{H}^0(X;\{U_a\}) \otimes H_0(X) \to \mathbb{Z}$$

which is a perfect pairing if X is a locally path connected space.

Exercise 1.1.7. *Show that if X is a smooth manifold, then there is an inclusion $\check{H}^0(X; \{U_a\}) \to H^0_{\mathrm{dR}}(X)$ given by tensoring with $\mathbb{R}$.*

Exercise 1.1.8. *We can replace $\mathbb{Z}$ by $\mathbb{Q}$ or $\mathbb{R}$ in the above construction (still using locally constant functions) and obtain the rational or real zeroth Čech cohomology groups. These are rational or real vector spaces. Show that if X is a smooth manifold then $\check{H}^0(X; \{U_a\}; \mathbb{R})$ is identified with $H^0_{\mathrm{dR}}(X)$.*

1.1.4 Zeroth Group Cohomology

Let G be a group, let A be an abelian group and let $\varphi : G \times A \to A$ be an action of G on A. We define $C^0(G; A)$, the group of zero-cochains for G with values in A to be A. We define $C^1(G; A)$, the group of one-cochains for G with values in A, to be the the set of all set functions $\psi : G \to A$. The one-cochains form an abelian group under addition of functions using the addition in A. We define $\delta : C^0(G; A) \to C^1(G; A)$ by $\delta(a)(g) = a - g \cdot a$. This is a group homomorphism. The kernel of this homomorphism is $H^0(G; A)$, the zeroth group cohomology of G with coefficients in A. Clearly, $H^0(G; A)$ is identified with A^G, the subgroup of A fixed pointwise by every $g \in G$.

As in the deRham case we have a cohomology group defined as the kernel of a 'coboundary' mapping.

If $f : K \to G$ is a homomorphism and the action of K on A is induced via f from the action of G on A, then there is an induced mapping $H^0(G; A) \to H^0(K; A)$ (any $a \in A$ which is G-invariant is automatically K-invariant). Thus, fixing A, $H^0(\cdot, A)$ is a contravariant functor from the category of groups G equipped with actions on A and homomorphisms compatible with the actions to the category of abelian groups.

Exercise 1.1.9. *Give an example of a connected space which is not path connected.*

Exercise 1.1.10. *For the space constructed in Exercise 1.1.9, show that the zeroth Čech cohomology is not dual to the zeroth singular homology.*

Exercise 1.1.11. *Show that for any locally path connected space the path components and the connected components agree and that the zeroth Čech cohomology is the dual of the zeroth singular homology.*

Exercise 1.1.12. *Prove that the zeroth singular homology is a functor from the category of topological spaces and continuous maps to the category of abelian groups.*

1.2 First Elements of Homological Algebra

The basic invariants described in the previous section are specific cases of much more general constructions. Before we begin discussing these constructions in detail, we need to develop some homological algebra that will be common to all of the invariants.

1.2.1 The Homology of a Chain Complex

A *chain complex* (C_*, ∂_*) consists of

- a graded abelian group, C_*, i.e. a set of abelian groups C_n indexed by the integers called the *chain groups*, and
- a homomorphism of graded groups, ∂_*, i.e. a set of homomorphisms, $\partial_n \colon C_n \to C_{n-1}$ again indexed by the integers, called the *boundary map*,

subject to the condition $\partial_{n-1} \circ \partial_n = 0$ for all n.

Often, we drop the index from the boundary homomorphisms and write the last condition as $\partial^2 = 0$.

Chain complexes form the objects of a category. The set of morphisms from C_* to D_* is the set of indexed homomorphisms $f_n \colon C_n \to D_n$ commuting with the boundary homomorphisms, i.e., satisfying $\partial_n \circ f_n = f_{n-1} \circ \partial_n$ where on the left-hand side of this equation the boundary map is the one from D_* whereas on the right-hand side the boundary map is the one from C_*.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The *homology* of a chain complex (C_*, ∂) is the graded abelian group $\{H_n(C_*)\}_n$ defined by

$$H_n = \frac{\mathrm{Ker}\partial_n \colon C_n \to C_{n-1}}{\mathrm{Im}\partial_{n+1} \colon C_{n+1} \to C_n}.$$

For each n, an element of $\mathrm{Ker}\,\partial \colon C_n \to C_{n-1}$ is called an *n-cycle*. An element in the image of $\partial \colon C_{n+1} \to C_n$ is called an *n-boundary*. When the degree is unimportant or obvious, we refer to these as cycles and boundaries respectively. An n-cycle ζ is said to be a cycle representative for a homology class $a \in H_n(C_*)$ if the equivalence class of ζ is a.

All the homology groups of a chain complex vanish if and only if the chain groups and boundary homomorphisms form a long exact sequence.

Homology is a functor from the category of chain complexes to the category of graded abelian groups (and homomorphisms). This last category is called the category of *graded abelian groups and homomorphisms*.

There is the dual notion of a cochain complex and its cohomology. In a cochain complex (C^*, δ) the upper index indicates the fact that the coboundary map δ raises degree by one. We require $\delta^2 = 0$, and the cohomology of (C^*, δ) is defined by

$$H^n(C^*) = \frac{\mathrm{Ker}(\delta) \colon C^n \to C^{n+1}}{\mathrm{Im}(\delta) \colon C^{n-1} \to C^n}.$$

Exercise 1.2.1. *Let (C_*, ∂_*) be a chain complex. Define (C^*, δ) by $C^n = C_{-n}$ and $\delta \colon C^n \to C^{n+1}$ by $\partial : C_{-n} \to C_{-n-1}$. Show that $H^n(C^*) = H_{-n}(C_*)$.*

1.2.2 Variants

The notion of chain complexes and homology and cochain complexes and cohomology exist in any abelian category. For example, for a commutative ring R one has the category of chain complexes consisting of R-modules and R-module homomorphisms. Homology is then a functor from this category to the category of sets of R-modules (and R-module homomorphisms) indexed by the integers. The special case when R is a field and everything in sight is then a vector space over the field is especially interesting.

Another variant is homology of a chain complex with coefficents. Suppose that A is an abelian group (e.g. $\mathbb{Z}/k\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$) and that (C_*, ∂) is a chain complex of abelian groups. We define $C_* \otimes A$ to be the chain complex of abelian groups whose n^{th}-chain group is $C_n \otimes_{\mathbb{Z}} A$ and whose boundary map $\partial\colon C_n \otimes_{\mathbb{Z}} A \to C_{n-1} \otimes_{\mathbb{Z}} A$ is $\partial \otimes \mathrm{Id}_A$. It is clear that this is a chain complex. We define $H_*(C_*; A)$ to be the homology of this chain complex. We say that $H_*(C_*; A)$ is the homology of (C_*, ∂) with coefficents in A. If A is a field then the homology groups are vector spaces over A, if A is a ring they are modules over A.

Exercise 1.2.2. *Show that if k is a field of characteristic zero then $H_*(C_* \otimes k) = H_*(C_*) \otimes k$*

Exercise 1.2.3. *Give an example of a chain complex of a free abelia groups whose homology is not free abelian.*

Exercise 1.2.4. *Consider the chain complex C_*,*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow 0$$

Show that $H_*(C_*) = 0$ but $H_*(C_* \otimes \mathbb{Z}/2) \neq 0$.

1.2.3 The Cohomology of a Chain Complex

If (C_*, ∂) is a chain complex, then we define the dual cochain complex (C^*, δ) by $C^n = \mathrm{Hom}(C_n, \mathbb{Z})$ with $\delta\colon C^n \to C^{n+1}$ the dual to $\partial\colon C_{n+1} \to C_n$. Clearly, $\delta^2 = 0$ so that we have defined a cochain complex. Its cohomology is called the cohomology of the original chain complex. More generally, if A is an abelian group then we can also define the cohomology of (C_*, ∂) with coefficents in A. To do this we define a cochain complex $C^*(A)$ by setting $C^n(A) = \mathrm{Hom}(C_n, A)$ and setting δ equal to the dual of ∂. Then the homology of this cochain complex, denoted $H^*(C_*; A)$, is the cohomology of C_* with coefficients in A.

Exercise 1.2.5. *For any abelian group A, compute the homology and cohomology of the chain complex in Exercise 1.2.4 with coefficients in A.*

1.2.4 The Universal Coefficient Theorem

Let G be an abelian group. A *short free resoution* of G is a short exact sequence,

$$0 \to K \to F \to G \to 0$$

where K and F are free abelian groups.

Lemma 1.2.6. *Every abelian group has a short free resoltuion.*

Proof. Every abelian group is the quotient of a free abelian group. Let F be that free abelian group, and K be the kernel of the quotient map $F \to G$. Then K is a subgroup of F and is thus also free abelian and we have the short free resolution,

$$0 \to K \to F \to G \to 0$$

as desired. $\square$

Now, let G and H be abelian groups. Take a short free resolution for G and consider $\mathrm{Hom}(\cdot, H)$. This may not give us a short exact sequence, but we do obtain an exact sequnce if we insert one more term, which we call $\mathrm{Ext}(G, H)$,

$$0 \to \mathrm{Hom}(G, H) \to Hom(F, H) \to Hom(K, H) \to \mathrm{Ext}(G, H) \to 0.$$

The group $Ext(G, H)$ is well defined up to canonical isomorphism. Suppose we have two short free resolutions for G, then a map $f : F_1 \to F_2$ exists so that the following diagram commutes,

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] \arrow[d, "f|_{K_1}"] & F_1 \arrow[r] \arrow[d, "f"] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_2 \arrow[r] & F_2 \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

This gives rise to,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(G, H) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(F_2, H) \arrow[r] \arrow[d, "f^*"] & \mathrm{Hom}(K_2, H) \arrow[r] \arrow[d, "(f|_{K_2})^*"] & E_2 \arrow[r] \arrow[d, "\exists! \bar{f}^*"] & 0 \\ 0 \arrow[r] & K_2 \arrow[r] & F_2 \arrow[r] & G \arrow[r] & 0 & \end{tikzcd}$$

We made a choice when we picked f. If we vary f, it becomes $f + h$ where $h : F_1 \to K_2$. This doesn't change $\bar{f}^*$.

Exercise 1.2.7. *Show that changing f to $f + h$ where $h : F_1 \to K_2$ doesn't change $\bar{f}^*$, and thus $E_1 \cong E_2$.*

We use a similar argument to construct the group $\mathrm{Tor}(G, H)$. Again start with two abelian groups G and H and a short free abelian resolution for G. Now, rather than looking at $\mathrm{Hom}(\cdot, H)$, we tensor with H. Again, this may to yeild a short exact sequence. Inorder to make an exact sequence we insert the extra term, $\mathrm{Tor}(G, H)$,

$$0 \to \mathrm{Tor}(G, H) \to K \otimes H \to F \otimes H \to G \otimes H \to 0.$$

Exercise 1.2.8. *Compute* $\mathrm{Tor}(G, H)$ *and* $\mathrm{Ext}(G, H)$ *for finitely generated abelian groups. Hint. First show that the construction behaves well under direct sum. Then compute for G and H cyclic.*

Exercise 1.2.9. *Show that if F is free abelian then $Ext(F, A) = 0$ and $Tor(F, A) = 0$ for any abelian group A.*

Exercise 1.2.10. *Let F be the free abelian group generated by $\{a_1, a_2, \ldots\}$. Define*

$$F \xrightarrow{f} \mathbb{Q} \longrightarrow 0$$

by $a_n \mapsto 1/n!$. Let $K = \ker\ f$. Find explicitly a free abelian basis for K. Use this to show that $Ext(\mathbb{Q}, \mathbb{Z})$ is uncountable.

Exercise 1.2.11. *Show that* $\mathrm{Tor}(A, \mathbb{Q}) = 0$ *for any finitely generated abelian group A.*

Exercise 1.2.12. *Show that* $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong A$ *for any finitely generated abelian group A.*

Exercise 1.2.13. *Show* $\mathrm{Tor}(\mathbb{Z}/n, A) = n$*-torsion of A for any abelian group A.*

Theorem 1.2.14. The Universal Coefficent Theorem *Let C_* be a free abelian chain complex and A an abelian group. Then there exist the following short exact sequences, natural for chain maps between such chain complexes,*

$$0 \to Ext(H_{k-1}(C_*); A) \to H^k(C_*; A) \to \mathrm{Hom}(H_k(C_*), A) \to 0$$

$$0 \to Ext(H_{k-1}(C_*); \mathbb{Z}) \to H^k(C_*) \to \mathrm{Hom}(H_k(C_*), \mathbb{Z}) \to 0$$

$$0 \to H_k(C_*) \otimes A \to H_k(C_*; A) \to \mathrm{Tor}(H_{k-1}(C_8); A) \to 0$$

Remark 1.2.15. These results are not true in general if C_* is not free abelian.

1.3 Basics of Singular Homology

The singular homology functor is a functor from the category of topological spaces to the category of graded abelian groups. It is the composition of two functors. The first is the singular homology chain complex functor which will be described in this section. The second is the homology functor applied to chain complexes as described in the preivous section.

1.3.1 The Standard n-simplex

Fix an integer $n \geq 0$, and let $\Delta^n \subset \mathbb{R}^{n+1}$ be the convex hull of the $n+1$ standard unit vectors $v_0 = (1, 0, \ldots, 0); v_1 = (0, 1, 0, \ldots, 0); \ldots, v_n = (0, 0, \ldots, 0, 1)$. The object Δ^n is called the *standard n-simplex*. We use affine coordinates on Δ^n. In these coordinates a point $x \in \Delta^n$ is represented by $(t_0, \ldots, t_n)$ subject to the conditions that $t_i \geq 0$ for all $0 \leq i \leq n$ and $\sum_{i=0}^n t_i = 1$. The point represented by the coordinates is $\sum_{i=0}^n t_i v_i$ in the affine structure on Δ^n. Notice that if A is an affine space and if we have points $a_0, a_1, \ldots, a_n$ of A then there is a unique affine linear map $\Delta^n \to A$ sending v_i to a_i for each $0 \leq i \leq n$. It sends the point with affine coordinates $(t_0, \ldots, t_n)$ to $\sum_{i=0}^n t_i a_i$. We call this *the affine linear map determined by the ordered set of points.*

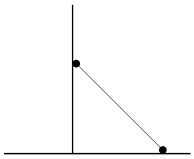

Figure 1: The standard one simplex, $\Delta^1 \subset \mathbb{R}^2$

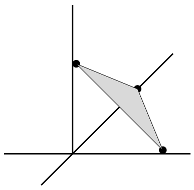

Figure 2: The standard two simplex, $\Delta^2 \subset \mathbb{R}^3$

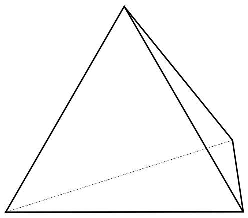

Figure 3: The three simplex, Δ^3. Note: This is **not** the standard embedding in $\mathbb{R}^4$!

Let X be a topological space. For $n \geq 0$ we define the group of singular n-chains in X, $S_n(X)$, to be the free abelian group generated by the set of all continuous maps $\sigma\colon \Delta^n \to X$. For $n < 0$ we define $S_n(X) = 0$. Associating to X the group $S_n(X)$ is a functor from the category of topological spaces and continuous maps to the category of groups and homomorphisms. The homomorphism $f_*\colon S_n(X) \to S_n(Y)$ associated to the continuous mapping $f\colon X \to Y$ sends $\sigma\colon \Delta^n \to X$ to $f_*\sigma = f \circ \sigma\colon \Delta^n \to Y$. This function from the basis of the free abelian group $S_n(X)$ to $S_n(Y)$ extends uniquely to a homomorphism $S_n(X) \to S_n(Y)$.

Our next goal is to provide a boundary map for this construction, so as to define a chain complex, *the singular chain complex*. As we shall see, to do this it suffices to define an element $\partial(\Delta^n) \in S_{n-1}(\Delta^n)$ for each $n \geq 0$. For each index i; $0 \leq i \leq n$ we define the i^{th}-face f_i of Δ^n. It is an affine linear map $f_i\colon \Delta^{n-1} \to \Delta^n$ determined by the n ordered points $\{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$. Notice that the image of f_i is the intersection of Δ^n with the hyperplane $t_i = 0$ in $\mathbb{R}^{n+1}$, and that f_i is an affine isomorphism onto this subspace preserving the order of the vertices.

We define

$$\partial(\Delta^n) = \sum_{i=0}^{n} (-1)^i f_i \in S_{n-1}(\Delta^n).$$

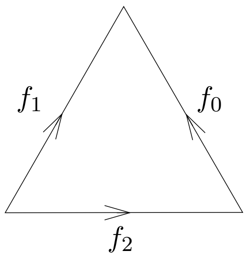

Figure 4: $\partial\Delta^2 = f_0 - f_1 + f_2$

More generally, let $\zeta = \sum_\sigma n_\sigma \sigma$ be an element of $S_n(X)$. (Here, our conventions are that σ ranges over the continuous maps of Δ^n to X, the n_σ are integers all but finitely many of which are zero.) Then we define

$$\partial(\zeta) = \sum_\sigma n_\sigma \sigma_*(\partial(\Delta^n)).$$

By functorality, the right-hand side of the above equality is naturally an element of $S_{n-1}(X)$.

Here is the basic lemma that gets this construction going. As we shall see, this will be the first of many similar computations.

Lemma 1.3.1. *The composition*

$$S_{n+1}(X) \xrightarrow{\partial} S_n(X) \xrightarrow{\partial} S_{n-1}(X)$$

is zero.

Proof. Clearly, by naturality, it suffices to prove that for all n we have $\partial(\partial(\Delta^n)) = 0 \in S_{n-2}(\Delta^n)$. For indices $0 \leq i < j \leq n$ let $f_{ij} \colon \Delta^{n-2} \to \Delta^n$ be the affine linear map that sends Δ^{n-2} isomorphically onto the intersection of Δ^n with the codimension-two subspace $t_i = t_j = 0$ in a manner preserving the ordering of the vertices. We compute:

$$\partial(f_i) = \sum_{k=i+1}^{n} (-1)^{k-1} f_{ik} + \sum_{k=0}^{i-1} (-1)^k f_{ki}.$$

Hence,

$$\partial\partial(\Delta^n) = \sum_{i=0}^{n} (-1)^i \Big(\sum_{k=i+1}^{n} (-1)^{k-1} f_{ik} + \sum_{k=0}^{i-1} (-1)^k f_{ki} \Big).$$

Claim 1.3.2. *In this sum, each term f_{ij} with $0 \leq i < j \leq n$ appears exactly twice and with cancelling signs.*

Exercise 1.3.3. *Prove this claim.*

□

The content of this lemma is that we have constructed a chain complex, $S_*(X)$. It is called the *singular chain complex* of X. It is immediate from what we have already proved and the definitions that this is a functor from the topological category to the category of chain complexes of free abelian groups.

The singular homology of X, denoted $H_*(X)$, is the homology of the singular chain complex. Singular homology is a functor from the topological category to the category of graded abelian groups and homomorphisms. The group indexed by n is denoted $H_n(X)$.

As disscussed in section 1.2.2 above, given a chain complex and an abelian group we can define the homology of the chain complex with coefficents in that abeian group. In particular, let A be an abelian group, then the singular homology of a topological space X with coefficients in A, denoted $H_*(X; A)$, is by definition the homology of $S_*(X) \otimes A$. Notice that if A is a field then these homology groups are vector spaces over A and if A is a ring, then they are modules over A. This is a functor from the category of topological spaces to the category of graded abelian groups (resp., to graded A-modules).

Likewise, we define the singular cohomology of X with coefficients in A to be the cohomology of the cochain complex $S^*(X; A) = \mathrm{Hom}(S_*(X), A)$. A case of particular interest is when $A = \mathbb{Z}$. In this case we refer to the cohomology as the singular cohomology of X.

1.3.2 First Computations

We will now compute some of the singular homology groups of a few spaces directly from the definition.

Lemma 1.3.4. *$H_i(X) = 0$ if $i < 0$.*

Proof. By definition $S_i(X) = 0$ if $i < 0$. The lemma follows immediately. □

Exercise 1.3.5. *Show for any A that the singular homology with coefficients in A vanishes in dimensions less than zero as does the singular cohomology with values in A.*

Since the singular zero chains are what we denoted by $S_0(X)$ earlier, and the singular one chains are generated by continuous maps of the interval into X, and since the boundary of the one chain represented by $\gamma\colon [0,1] \to X$ is $\gamma(1) - \gamma(0)$, it follows that $H_0(X)$ is the quotient of $S_0(X)$ by the equivalence relation studied earlier. Hence, we have already established the following result:

Lemma 1.3.6. *$H_0(X)$ is the free abelian group generated by the set of path components of X.*

Exercise 1.3.7. *Show that the singular cohomology of a topological space X is $\mathbb{Z}^{\mathcal{P}(X)}$, the group of functions from the set of path components of X to the integers.*

1.3.3 The Homology of a Point

Proposition 1.3.8. *Let X be a one-point space. Then $H_k(X) = 0$ for all $k \neq 0$ and $H_0(X) \cong \mathbb{Z}$.*

Proof. For each $n \geq 0$, there is exactly one map of $\Delta^n \to X$, let us call it σ_n. Thus, $S_n(X) = \mathbb{Z}$ for every $n \geq 0$. Furthermore, $\partial(\sigma_n) = \sum_{i=0}^{n}(-1)^i \sigma_{n-1}$. This sum is zero if n is odd and is σ_{n-1} if n is even and greater than zero. Hence, the group of n-cycles is trivial for n even and greater than zero, and is $\operatorname{Ker} \partial_n = \mathbb{Z}$ for n odd and for $n = 0$. On the other hand, the group of n boundaries is trivial for n even and all of $S_n(X)$ for n odd. The result follows immediately. □

Exercise 1.3.9. *Compute the singular cohomology of a point. For any abelian group A, compute the singular homology and cohomology with coefficients in A for a point.*

1.3.4 The Homology of a Contractible Space

We say that a space X is *contractible* if there is a point $x_0 \in X$ and a continuous mapping $H\colon X \times I \to X$ with $H(x,1) = x$ and $H(x,0) = x_0$ for all $x \in X$. As an example, $\mathbb{R}^n$ is contractible for all $n \geq 0$.

Exercise 1.3.10. *Show $\mathbb{R}^n$ is contractible.*

Exercise 1.3.11. *Show any convex subset $\mathbb{R}^n$ is contractible.*

Proposition 1.3.12. *If X is contractible, then $H_i(X) = 0$ for all $i \geq 1$.*

Proof. Let H be a contraction of X to x_0. Let $\sigma\colon \Delta^k \to X$ be a continuous map. We define $c(\sigma)\colon \Delta^{k+1} \to X$ by coning to the origin. Thus,

$$c(\sigma)(t_0, \ldots, t_{k+1}) = (1 - t_0)\sigma(t_1/(1 - t_0), \ldots, t_{k+1}/(1 - t_0)).$$

One checks easily that this expression makes sense and that as $t_0 \to 1$ the limit exists and is x_0, so that the map is well-defined and continuous on the entire Δ^{k+1}.

Claim 1.3.13. *If $k \geq 1$, then $\partial(c(\sigma)) = \sigma - \sum_{i=0}^{k}(-1)^k c(\sigma \circ f_i)$.*

This claim remains true for $k = 0$ if we interpret the cone on the empty set to be the origin.

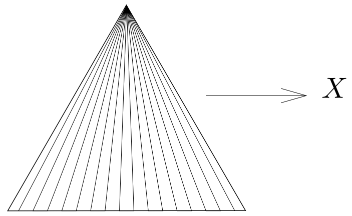

Figure 5: $c(\sigma : \Delta^1 \to X) : \Delta^2 \to X$

For all $k \geq 0$, we define $c\colon S_k(X) \to S_{k+1}(X)$ by $c(\sum n_\sigma \sigma) = \sum n_\sigma c(\sigma)$. Then claim 1.3.13 implies the following fundamental equation for all $\zeta \in S_k(X),\ k \geq 1$:

$$\partial c(\zeta) = \zeta - c(\partial \zeta) \tag{1}$$

It follows that if ζ is a k cycle for any $k \geq 1$, then $\partial c(\zeta) = \zeta$, and hence the homology class of ζ is trivial. Thus, $H_k(X) = 0$ for all $k > 0$. Clearly, we have $H_k(X) = 0$ for $k < 0$ and since X is path connected, we have $H_0(X) \cong \mathbb{Z}$. This completes the proof of the proposition. □

Exercise 1.3.14. *Compute $H_0(X)$ for X contractible.*

Exercise 1.3.15. *For any abelian group A, compute the homology and cohomology of a contractible space with values in A.*

1.3.5 Nice Representative One-cycles

Before we begin any serious computations, we would like to give a feeling for the kind of cycles that can be used to represent one-dimensional singular homology. Let X be a path connected space and let $a \in H_1(X)$ be a homology class. Our object here is to find an especially nice cycle representative for a.

Definition 1.3.16. A *circuit* in X is a finite ordered set of singular one simplices $\{\sigma_1, \ldots, \sigma_k\}$ in X with the property that $\sigma_i(1) = \sigma_{i+1}(0)$; for $1 \leq i \leq k-1$ and $\sigma_k(1) = \sigma_1(0)$.

Given a circuit $\{\sigma_1, \ldots, \sigma_k\}$ there is an associated singular one-chain $\zeta = \sum_{i=1}^{k} \sigma_i \in S_1(X)$.

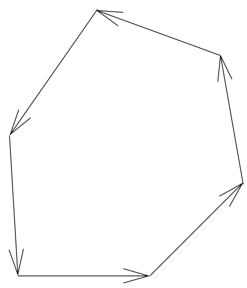

Figure 6: A circuit

Exercise 1.3.17. *Show that ζ is a cycle.*

Hence ζ represents a homology class $[\zeta] \in H_1(X)$. Any such homology class is the image under a continuous map of a homology class in the circle. To see this, paste k copies of the one-simplex together end-to-end in a circuit to form a circle T, and use the σ_i, in order, to define a map $\tilde{\zeta}\colon T \to X$. The inclusions of the k unit intervals into T form a singular one-cycle $\mu \in S_1(T)$ representing a homology class $[\mu] \in H_1(T)$. Clearly, $[\zeta] = \tilde{\zeta}_*[\mu]$. The next proposition shows that all elements in $H_1(X)$ are so represented, at least when X is path connected.

Proposition 1.3.18. *Let X be path connected. Given $a \in H_1(X)$, there is a circuit $\{\sigma_1, \ldots, \sigma_k\}$ such that the singular one-cycle $\zeta = \sigma_1 + \cdots + \sigma_k$ represents the class a.*

Proof. We sketch the proof and leave the details to the reader as exercises.

Lemma 1.3.19. *Every homology class in $H_1(X)$ is represented by a cycle all of whose non-zero coefficients are positive.*

Proof. First let us show that we can always find a representative one-cycle such that the only singular one simplices with negative coefficients are constant maps. Given $\sigma\colon \Delta^1 \to X$, written as $\sigma(t_0, t_1)$, we form the map $\psi\colon \Delta^2 \to X$ by $\psi(t_0, t_1, t_2) = \sigma(t_0 + t_2, t_1)$. It is easy to see that $\partial\Delta = \sigma + \tau - p$ where $\tau(t_0, t_1) = \sigma(t_1, t_0)$ and $p(t_0, t_1) = \sigma(1, 0)$. We can rewrite this as an equivalence

$$-\sigma \cong \tau - p.$$

Using this relation allows us to remove all negative coefficients from non-trivial singular one-simplices at the expense of introducing negative coefficients on constant singular one-simplices.

Next we consider a constant map $p\colon \Delta^2 \to X$. Its boundary is a constant singular one simplex at the same point. Subtracting and adding these relations allows us to remove all constant one simplices (with either sign) from the cycle representative for a. $\square$

Corollary 1.3.20. *Every homology class in $H_1(X)$ is represented by a sum of maps of Δ^1 to X with coefficients one (but possibly with repetitions).*

Now let $\zeta = \sum_{i=1}^{k} \sigma_i$ be a cycle. It is possible to choose a subset, such that after possibly reordering, we have a circuit $\{\sigma_{i_1}, \ldots, \sigma_{i_\ell}\}$ Continuing in this way we can write ζ as a finite sum of singular cycles associated to circuits.

Our next goal is to combine two circuits into a single one (after modifying by a boundary). Let ζ and ζ' be cycles associated with circuits $\{\sigma_1, \ldots, \sigma_k\}$ and $\{\sigma'_1, \ldots, \sigma'_\ell\}$. Since X is path connected, there is a path $\gamma\colon [0,1] \to X$ connecting $\sigma_1(0)$ to $\sigma'_1(0)$. Then

$$\{\sigma_1, \ldots, \sigma_k, \gamma, \sigma'_1, \ldots, \sigma'_\ell, \gamma^{-1}\}$$

is a circuit and we claim that its associated cycle is homologous to the sum of the cycles associated to the two circuits individually. (Here, $\gamma^{-1}(t_0, t_1) = \gamma(t_1, t_0)$.)

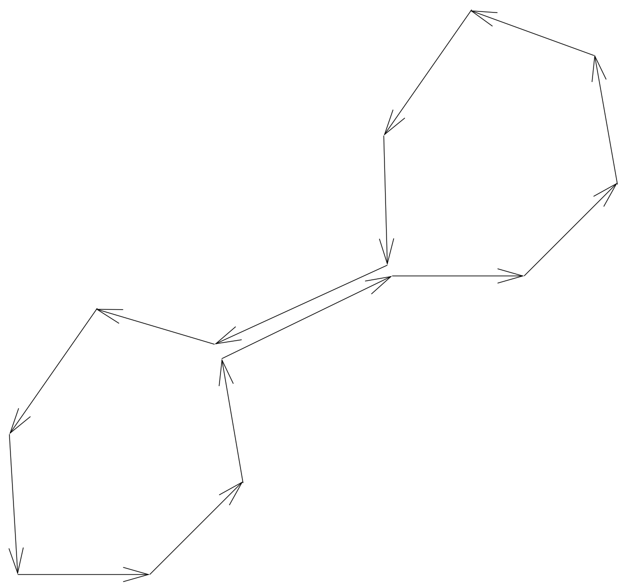

Figure 7: Joining two circuits together

Exercise 1.3.21. *Prove this last statement.*

An inductive argument based on this construction then completes the proof of the proposition. □

1.3.6 The First Homology of S^1

The purpose of this section is to make our first nontrivial computation – that of H_1 of the circle. We denote by S^1 the unit circle in the complex plane. It is a group and we write the group structure multiplicatively. Here is the result.

Theorem 1.3.22.

$$H_1(S^1) \cong \mathbb{Z}.$$

The proof of this result is based on the idea of winding numbers.

Claim 1.3.23. *Let I be the unit interval, and let $f: I \to S^1$ be a continuous mapping. Then there is a continuous function $\theta: I \to \mathbb{R}$ such that for all $t \in I$ $\exp(i\theta(t)) = f(t)$. The function θ is unique up to adding a constant integral multiple of 2π. In particular, $\theta(1) - \theta(0)$ is independent of θ. This difference is called the* ***winding number*** *$w(f)$ of f. Furthermore, $\exp(iw(f)) = f(1)f(0)^{-1}$.*

Proof. Let us consider uniqueness first. Suppose that θ and θ' are functions as in the claim. Then $\exp(i\theta(t)) = \exp(i\theta'(t))$ for all $t \in I$. Hence, $\theta(t) - \theta'(t)$ is an integral multiple of 2π. By continuity, this multiple is constant as we vary t.

Now we turn to existence. We consider the set $\mathcal{S} = \{t \in I\}$ for which a function θ as required exists for the subinterval $[0, t]$. Clearly, $0 \in \mathcal{S}$. Suppose that $t \in \mathcal{S}$, and let $\theta_0: [0, t] \to \mathbb{R}$ be as required. There exists an open subset $U \subset I$ containing t such that $f(U) \subset S^1 \setminus \{-f(t)\} =: T$. There is a continuous function $\theta_T: T \to \mathbb{R}$ such that $\exp(i\theta_T(x)) = x$ for all $x \in T$. By adding an integral multiple of 2π we can assume that $\theta_T(f(t)) = \theta_0(t)$. Defining θ' to be θ on $[0, t]$ and to be $\theta_T \circ f$ on U gives a function as required on $[0, t] \cup U$. This shows that $\mathcal{S}$ is an open subset of I. To show that $\mathcal{S}$ is closed suppose that we have a sequence $t_n \to t$ and functions $\theta_n: [0, t_n] \to \mathbb{R}$ as required. By adjusting by integral multiples of 2π we can arrange that all these functions take the same value at 0. Then by uniqueness, we see that θ_n and θ_m agree on their common domain of definition. Thus together they define a function of $\cup_{n=1}^{\infty}[0, t_n]$ which surely includes $[0, t)$. It is easy to see that the limit as $n \mapsto \infty$ of $\theta(t_n)$ exists and hence can be used to extend the map to a continuous map on $[0, t]$, as required. To see this, look at an interval $U \subset S^1$ containing $f(t)$. Now, consider the preimage of U in $\mathbb{R}$ under the exponential map $x \mapsto \exp ix$. This is a disjoint union of intervals, $\amalg U_i$ mapping homeomorphically to U, and we have $\theta((t - \epsilon, t)) \subset U_i$ for some i. Now, it should be clear that the limit exists, and we define $\theta(t) = \underset{t_n \to t}{lim}\, \theta(t_n)$. Since $\mathcal{S}$ is open and closed and contains 0, it follows that $\mathcal{S} = I$ and hence $1 \in \mathcal{S}$, which proves the result. $\square$

In fact the argument above can be used to show the following

Corollary 1.3.24. *Suppose that f, g are close maps of I to S^1, close in the sense that the angle between $f(t)$ and $g(t)$ is uniformly small. Then there are functions θ_f and θ_g from $I \to \mathbb{R}$ as above which are close, and in particular $w(f)$ and $w(g)$ are close.*

For a singular 0-chain $\mu = \sum_p n_p p$ in S^1, we define $\theta(\mu) = \prod_p p^{n_p}$, using the multiplicative group structure of S^1.

Lemma 1.3.25. *Let $\zeta = \sum_{i=1}^k n_i \sigma_i$ be a singular one-chain in S^1. Define $w(\zeta) = \sum_{i=1}^k n_i w(\sigma_i)$. Then $\exp(iw(\zeta)) = \theta(\partial\zeta)$.*

Proof. By the multicative property of both sides, it suffices to prove this equality for a singular one-simplex, where it is clear. □

Corollary 1.3.26. *If ζ is a singular one-cycle in S^1, then $w(\zeta) \in 2\pi\mathbb{Z}$.*

We define a function from the abelian group of one cycles to $\mathbb{Z}$ by associating to a cycle ζ the integer $w(\zeta)/2\pi$. This function is clearly additive and hence determines a homomorphism w from the group of one-cycles to $\mathbb{Z}$.

Claim 1.3.27. *If ζ is a boundary, then $w(\zeta) = 0$.*

Proof. Again by multiplicativity, it suffices to show that if $\zeta = \partial\tau$ where $\tau\colon \Delta^2 \to S^1$ is a singular two simplex, then $w(\zeta) = 0$. To establish this we show that there is a continuous function $\theta\colon \Delta^2 \to \mathbb{R}$ such that $\exp(i\theta(x)) = \tau(x)$ for all $x \in \Delta^2$. First pick a lifting $\theta(1,0,0) \in \mathbb{R}$ for $\tau(1,0,0) \in S^1$. Now for each (a,b) with $a, b \geq 0$ and $a + b = 1$, consider the interval $\{t_0, (1-t_0)a, (1-t_0)b\}$. There is a unique lifting $\theta_{a,b}$ mapping this interval into $\mathbb{R}$ lifting the restriction of τ to this interval and having the given value at $(1,0,0)$. The continuity property described above implies that these lifts fit together to give the map $\theta\colon \Delta^2 \to \mathbb{R}$ as required. Once we have this map we see that $w(f_0) = \theta(0,0,1) - \theta(0,1,0)$, $w(f_1) = \theta(0,0,1) - \theta(1,0,0)$ and $w(f_2) = \theta(0,1,0) - \theta(1,0,0)$, so that $w(f_0) - w(f_1) + w(f_2) = 0$. □

Exercise 1.3.28. *Prove that the map θ constructed above is continuous.*

It now follows that we have a homomorphism $W\colon H_1(S^1) \to \mathbb{Z}$. We shall show that this map is an isomorphism. The map $I \to S^1$ given by $t \mapsto \exp(2\pi it)$ is a one cycle whose image under w is 1. This proves that W is onto. It remains to show that it is one-to-one. Suppose that ζ is a cycle and $w(\zeta) = 0$. According to proposition 1.3.18, we may as well assume that ζ is the cycle associated to a circuit $\{\sigma_1, \ldots, \sigma_k\}$. We choose liftings $\tilde{\sigma}_\ell\colon I \to \mathbb{R}$ with $\exp(i\tilde{\sigma}_\ell(t)) = \sigma_\ell(t)$ for all ℓ and all $t \in I$. We do this in such a way that $\tilde{\sigma}_{\ell+1}(0) = \sigma_\ell(1)$ for $1 \leq \ell < k$. The difference $\tilde{\sigma}_k(1) - \tilde{\sigma}_1(0)$ is $2\pi w(\zeta)$, which by assumption is zero. This implies that the circuit in S^1 lifts to a circuit in $\mathbb{R}$. Let $\tilde{\zeta}$ be the one cycle in $\mathbb{R}$ associated to this circuit. By Proposition 1.3.12 there is a two-chain $\tilde{\mu}$ in $\mathbb{R}$ with $\partial\tilde{\mu} = \tilde{\zeta}$. Let μ be the two-chain in S^1 that is the image of $\tilde{\mu}$ under the exponential mapping. We have $\partial(\mu) = \zeta$, and hence that ζ represents the trivial element in homology. This completes the proof of the computation of $H_1(S^1)$.

Exercise 1.3.29. *Complete the proof of Corollary 1.3.20*

Exercise 1.3.30. *State and prove the generalization of Proposition 1.3.18 in the case of a general space X.*

Exercise 1.3.31. *Prove Claim 1.3.13*

Exercise 1.3.32. *Show that if X is path connected then for any $a \in H_1(X)$, there is a map $f\colon S^1 \to X$ and an element $b \in H_1(S^1)$ such that $a = f_*(b)$.*

Exercise 1.3.33. *Let X be path connected and simply connected (which means that every continuous map of $S^1 \to X$ extends over D^2). Show that $H_1(X) = 0$.*

1.4 An Application: The Brouwer Fixed Point Theorem

As an application we now prove a famous result. Recall that a *retraction* of a space X onto a subspace $Y \subset X$ is a continuous surjective map $\varphi : X \to Y$ such that $\varphi|_Y = \mathrm{Id}_Y$.

Theorem 1.4.1. *There is no continuous retraction of the 2-disk D^2 onto its boundary S^1.*

Proof. Let $i\colon S^1 \to D^2$ be the inclusion of S^1 as the boundary of D^2. Suppose a continuous retraction $\varphi : D^2 \to S^1$ of the disk onto its boundary exists. Then the composition

$$S^1 \xrightarrow{i} D^2 \xrightarrow{\varphi} S^1,$$

is the identity. Since homology is a functor we have

$$H_1(S^1) \xrightarrow{i_*} H_1(D^2) \xrightarrow{\varphi_*} H_1(S^1)$$

is the identity. But since $H_1(D^2) = 0$ since the disk is contractible, and $H_1(S^1) \neq 0$, this is impossible. Hence, no such φ exists. □

This leads to an even more famous result.

Theorem 1.4.2. The Brouwer Fixed Point Theorem *Any continuous map of the two disk, D^2, to itself has a fixed point.*

Proof. Suppose that $\psi\colon D^2 \to D^2$ is a continuous map without a fixed point. Then, the points x and $\psi(x)$ are distinct points of the disk. The line $L(x)$ passing through x and $\psi(x)$ meets the boundary of the disk S^1 in two points. Let $\varphi(x)$ be the point of $L(x) \cap S^1$ that lies on the open half-ray of this line beginning at $\psi(x)$ and containing x. One sees easily that $L(x)$ and $\varphi(x)$ vary continuously with x. Thus, $\varphi(x)$ is a continous mapping $D^2 \to S^1$. Also, it is clear that if $x \in S^1$, then $\varphi(x) = x$. This contradicts theorem 1.4.1 and concludes the proof of the Brouwer fixed point theorem. □

Exercise 1.4.3. *Show that φ is continuous.*

Remark 1.4.4. This result was proved before the introduction of homology by direct topological arguments; however, these are quite intricate. The fact that this result follows so easily once one has the machinery of homology was one of the first indications of the power of homology.

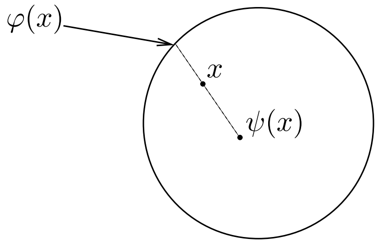

Figure 8: $\varphi : D^2 \to S^1$

2 The Axioms for Singular Homology and Some Consequences

As the above computations should indicate, it is quite difficult to compute singular homology directly from the definition. Rather, one proceeds by finding general properties that homology satisfies that allow one to compute homology for a large class of spaces from a sufficiently detailed topological description of the space. These general properties are called the *axioms for homology*. One can view this in two ways – from one point of view it is a computational tool that allows one to get answers. From a more theoretical point of view, it says that any homology theory that satisfies these axioms agrees with singular homology, no matter how it is defined, at least on the large class of spaces under discussion.

2.1 The Homotopy Axiom for Singular Homology

The result in section 1.3.4 that the homology groups of a contractible space are the same as those of a point has a vast generalization, the *Homotopy Axiom* for homology.

Definition 2.1.1. Let X and Y be topological spaces. We say two maps $f, g\colon X \to Y$ are *homotopic* if there is a continuous map $F\colon X \times I \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. The map F is called a *homotopy* from f to g.

Lemma 2.1.2. *Homotopy is an equivalence relation on the set of maps from X to Y.*

Exercise 2.1.3. *Prove this lemma.*

An equivalence class of maps under this equivalence relation is called a *homotopy class*. We will denote the homotopy class of a map f by $[f]$, and f is homotopic to g by $f \simeq g$. With this terminology in hand we can define a category, ***Homotopy***. The objects of this category are topological spaces. If X and Y are two topological spaces, then the collection of morphisms in ***Homotopy*** between X and Y are just the homotopy classes of maps from X to Y. That is, $Hom(X,Y) = \{f : X \to Y | f \text{ is continuous}\}/\simeq$. So the morphisms in this category are actually equivalence classes of maps. If $[f] : X \to Y$ and $[g] : Y \to Z$ are two homotopy classes of maps, their composition $[g] \circ [f] : X \to Z$ in ***Homotopy*** is the class $[g \circ f]$. We need to check that this definition is well defined. Suppose $f' \in [f]$ is another representative for the homotopy class of f. Let $H : X \times I \to Y$ be a homotopy from f to

f'. Then the map $H' = g \circ H : X \times I \to Z$ is continuous since both H and g are continuous and we have $H'(x, 0) = g \circ H(x, 0) = g \circ f$ and $H'(x, 1) = g \circ H(x, 1) = g \circ f'$. So H' is a homotpy from $g \circ f$ to $g \circ f'$, and thus $[g \circ f] = [g \circ f']$. A similar argument shows that the defintion is independent of the representative we choose for $[g]$. For any object X of ***Homotopy***, we have the identity morphism, $[\mathrm{Id}_X] : X \to X$ is just the hompotopy class represented by the identity map on X. Similarly, the morphisms in this category are associative under composition because the underlying representatives of the homotopy classes are associative. Notice that in the category ***Homotopy***, it doesn't make sense to ask for the value of a morphism at a point in X. Unlike the morphisms in many other categories we have seen, such as the algebraic categories of groups or rings, there is no specific set function underlying a morphism in this category. Also, notice that there is a natural functor from the category ***Top*** of topological spaces and continuous maps to the category ***Homotopy*** given by sending a topological space to itself and sending a continuous function to the homotopy class represented by that function.

We say that a map $f \colon X \to Y$ is a homotopy equivalence if there is a map $g \colon Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y. Show that the relation X is homotopy equivalent to Y is an equivalence relation.

Recall, (See A) in any category $\mathcal{C}$ a morphism $f : a \to b$ is called an *isomorphism* if there is another morphism $f' : b \to a$ in $\mathcal{C}$ such that $f' \circ f = \mathrm{Id}_a$ and $f \circ f' = \mathrm{Id}_b$. If such a morphism f' exists, it is unique and we write $f' = f^{-1}$. Two objects a and b are said to be *isomorphic* if there exists an isomorphism between them.

Exercise 2.1.4. *What are the isomorphisms in the category **Homotopy**?*

Lemma 2.1.5. *Let X be a space and $x_0 \in X$ a point. Then the inclusion of x_0 into X is a homotopy equivalence if and only if X is contractible to x_0.*

Exercise 2.1.6. *Prove this lemma.*

Now we are ready to state the vast generalization alluded to above.

Theorem 2.1.7. The Homotopy Axiom for Singular Homology *Let X and Y be topological spaces and let $f, g \colon X \to Y$. If f and g are homotopic, then $f_* = g_* \colon H_*(X) \to H_*(Y)$.*

The Homotopy Axiom is most frequently used in the form of the following immediate corollary.

Corollary 2.1.8. *If $f \colon X \to Y$ is a homotopy equivalence, then $f_* \colon H_n(X) \to H_n(Y)$ is an isomorphism for every n.*

In the language of categories, this corollary to the homotopy axiom says that the singular homology functor from the category of topological spaces to the category of graded abelian groups factors through the natural functor from ***Top*** to ***Homotopy***.

Proof. First let us introduce the homological analogue of a homotopy.

Definition 2.1.9. Let C_* and D_* be chain complexes and $f_*, g_* \colon C_* \to D_*$ be morphisms in the category of chain complexes. A *chain homotopy* from f_* to g_* is a collection of homomorphisms, one for each integer k, $H_k \colon C_k \to D_{k+1}$ satisfying

$$\partial \circ H_k + H_{k-1} \circ \partial = g_* - f_*.$$

Claim 2.1.10. *Chain homotopy is an equivalance relation on morphisms from C_* to D_*.*

Exercise 2.1.11. *Prove this claim.*

Claim 2.1.12. *If $f_*, g_* \colon C_* \to D_*$ are chain homotopic, then the maps that they induce on homology are equal.*

Proof. If $\zeta \in C_k$ is a cycle, then,

$$\partial(H_k(\zeta)) = g_*(\zeta) - f_*(\zeta).$$

□

Thus, in order to prove the Homotopy Axiom it is sufficent to prove the following proposition:

Proposition 2.1.13. *If $f, g \colon X \to Y$ are homotopic, then $f_*, g_* \colon S_*(X) \to S_*(Y)$ are chain homotopic.*

Proof. Since $\Delta^k \times I$ is a product of affine spaces and hence is itself affine, we can define an affine map of a simplex Δ^n into it by simply giving an ordered set of $n+1$ points in the space. We denote the affine map of Δ^n into $\Delta^k \times I$ determined by $(x_0, \ldots, x_n)$ by this ordered $n+1$-tuple. We denote by u_i the point $(v_i, 0)$ in $\Delta^k \times I$ and by w_i the point $(v_i, 1)$. Let $H(\Delta^k) \in S_{k+1}(\Delta^k \times I)$ be defined by

$$H(\Delta^k) = \sum_{i=0}^{k} (-1)^i (u_0, \ldots, u_i, w_i, \ldots, w_k).$$

Claim 2.1.14.

$$\partial(H(\Delta^k)) = \Delta^k \times \{1\} - \Delta^k \times \{0\} - \sum_{i=0}^{k} (-1)^k (f_i \times \mathrm{Id}_I)_*(H(\Delta^{k-1})).$$

Proof.
Being careful that the sign corresponds with the position in the (n+2)-tuple of the dropped vertex, rather than its index, we see that the boundary formula yields, $\partial(H(\Delta^k)) =$

$$\sum_{i=0}^{k}(-1)^i\Big[\sum_{j=0}^{i}(-1)^j(u_0,\ldots,\hat{u}_j,\ldots,u_i,w_i,\ldots,w_k)+\sum_{j=i}^{k}(-1)^{j+1}(u_0,\ldots,u_i,w_i,\ldots,\hat{w}_j,\ldots,w_k)\Big]=$$
$$\sum_{j<i}(-1)^{i+j}(u_0,\ldots,\hat{u}_j,\ldots,u_i,w_i,\ldots,w_k)+\sum_{j>i}(-1)^{i+j+1}(u_0,\ldots,u_i,w_i,\ldots,\hat{w}_j,\ldots,w_k)+$$
$$\sum_{i=0}^{k}\big[(u_0,\ldots,u_{i-1},w_i,\ldots,w_k)-(u_0,\ldots,u_i,w_{i+1},\ldots,w_k)\big].$$

The last term telescopes to leave only,

$$(w_0,\ldots,w_k)-(u_0,\ldots,u_k)=\Delta^k\times\{1\}-\Delta^k\times\{0\},$$

and so we obtain,

$$\begin{aligned}\partial(H(\Delta^k)) &= \sum_{j<i}(-1)^{i+j}(u_0,\ldots,\hat{u}_j,\ldots,u_i,w_i,\ldots,w_k) \\ &+\sum_{j>i}(-1)^{i+j+1}(u_0,\ldots,u_i,w_i,\ldots,\hat{w}_j,\ldots,w_k) \\ &+\Delta^k\times\{1\}-\Delta^k\times\{0\}.\end{aligned} \tag{2}$$

Now consider $\Delta^{k-1}\times I$. Let $x_0,\ldots,x_{n-1}$ denote the vertices at 0, and $y_0,\ldots,y_{n-1}$ denote the vertices at 1. Then,

$$(f_j\times\mathrm{Id}_I)(x_j)=\begin{cases}u_i & j>i\\ u_{i+1} & j\leqslant i\end{cases}$$

$$(f_j\times\mathrm{Id}_I)(y_j)=\begin{cases}w_i & j>i\\ w_{i+1} & j\leqslant i\end{cases}$$

Now, we consider $H(\partial\Delta^k)$. It is defined to be,

$$\sum_{j=0}^{k}(-1)^j(f_j\times\mathrm{Id}_I)(H(\Delta^{k-1})=\sum_{j=0}^{k}(-1)^j(f_j\times\mathrm{Id}_I)(\sum_{i=0}^{k-1}(-1)^i(x_0,\ldots,x_i,y_i,\ldots,y_k))=$$
$$\sum_{j=0}^{k}(-1)^j\Big[\sum_{i=0}^{j-1}(-1)^i(u_0,\ldots,u_i,w_i,\ldots,\hat{w}_j,\ldots,w_k)+\sum_{i=0}^{j-1}(-1)^{i-1}(u_0,\ldots,\hat{u}_j,w_{i+1},\ldots,\ldots,w_k)\Big]=$$
$$\sum_{j>i}(-1)^{i+j}(u_0,\ldots,u_i,w_i,\ldots,\hat{w}_j,\ldots,w_k)+\sum_{j<i}(-1)^{i+j-1}(u_0,\ldots,\hat{u}_j,\ldots,u_i,w_i,\ldots,w_k)=$$
$$-\Big[\sum_{j>i}(-1)^{i+j+1}(u_0,\ldots,u_i,w_i,\ldots,\hat{w}_j,\ldots,w_k)+\sum_{j<i}(-1)^{i+j}(u_0,\ldots,\hat{u}_j,\ldots,u_i,w_i,\ldots,w_k)\Big]$$

Comparing to the formula (2) we found above for $\partial(H(\Delta^k))$, we see that we have,

$$\partial(H(\Delta^k)) = \Delta^k \times \{1\} - \Delta^k \times \{0\} - \sum_{i=0}^{k}(-1)^k(f_i \times \mathrm{Id}_I)_*(H(\Delta^{k-1})).$$

$\square$

Suppose that $F\colon X \times I \to Y$ is a homotopy from f to g. Then we define $F_*\colon S_k(X) \to S_{k+1}$ by

$$F_*(\sigma) = F \circ (\sigma \times \mathrm{Id}_I)_*(H(\Delta^k)).$$

Claim 2.1.15. *F_* is a chain homotopy from f_* to g_*.*

Proof.

$$\partial F_*(\sigma) = (F \circ (\sigma \times \mathrm{Id}_I)_*(\Delta^k \times \{1\} - \Delta^k \times \{0\} - H(\partial\Delta^k)) = g_*(\sigma) - f_*(\sigma) - F_*(\partial\sigma).$$

So,

$$\partial F_*(\sigma) + F_*(\sigma) = g_*(\sigma) - f_*(\sigma),$$

and thus F_* is a chain homotopy from f to g. $\square$

The proposition now follows immediately. $\square$

This completes the proof of the homotopy axiom. In brief, these were the main steps:

Step 1. Chain homotopic maps induce the same map on homology.

Step 2. Define an element $H(\Delta^k) \in S_{k+1}(\Delta^k \times I)$ with the property

$$\partial(H(\Delta^k)) = \Delta^k \times \{1\} - \Delta^k \times \{0\} - \sum_{i=0}^{k}(-1)^k(f_i \times \mathrm{Id}_I)_*(H(\Delta^{k-1})).$$

Step 3. Given a homotopy F from f to g use $H(\Delta^k)$ to define a chain homotopy F_* from f_* to g_* by

$$F_*(\sigma) = F \circ (\sigma \times \mathrm{Id}_I)_*(H(\Delta^k)).$$

$\square$

2.2 The Mayer-Vietoris Theorem for Singular Homology

The next axiom is a result that allows us to compute the homology of a union of two open sets provided that we know the homology of the sets themselves and the homology of their intersection.

Theorem 2.2.1. *Let X be a topological space and U, V open subsets of X so that $X = U \cup V$. Let $j_U : U \hookrightarrow X$ and $j_V : V \hookrightarrow X$ and $i_U : U \cap V \hookrightarrow U$ and $i_V : U \cap V \hookrightarrow V$ be the inclusions. Then there is a long exact sequence of homology groups:*

$$\cdots \to H_k(U \cap V) \xrightarrow{(i_U)_* - (i_V)_*} H_k(U) \oplus H_k(V) \xrightarrow{(j_U)_* + (j_V)_*} H_k(X) \longrightarrow H_{k-1}(U \cap V) \to \cdots$$

In order to prove this we will need the following homological lemma:

Lemma 2.2.2. *Suppose we have a short exact sequence of chain complexes:*

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0;$$

that is, we have three chain complexes, A_*, B_*, C_* with chain complex morphisms $f_* : A_* \to B_*$ and $g_* : B_* \to C_*$ such that for every n we have a short exact sequence:

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

Then there exists a connecting homomorphism $\beta : H_k(C_*) \to H_{k-1}(A_*)$ making the following long exact sequence in homology:

$$\cdots \longrightarrow H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\beta} H_{n-1}(A_*) \longrightarrow \cdots$$

Proof. The proof is an exercise in what is known as diagram chasing. Diagram chasing is very common in algebraic topology, but it is a technique that is extremely tedious to read and only becomes comfortable with practice. We will write out explicitly the begining of the proof and leave the rest to the reader. To begin we will prove exactness at $H_n(B_*)$. Exactness at this point involves only the maps on homology induced by f_* and g_*, not the connecting homomorphism. First, we have $g_* \circ f_* = 0$ holds on the chain level, so it is also true on homology. Now, let $[\zeta] \in H_n(B_*)$. Suppose that $g_*[\zeta] = 0$. This implies $g_*(\zeta) = \partial c_{n+1}$ for some $c_{n+1} \in C_{n+1}$. By exactness at C_{n+1}, there is a $b_{n+1} \in B_{n+1}$ that maps to c_{n+1} under g_{n+1}. Let $\zeta' = \zeta - \partial b_{n+1}$. Then ζ' is clearly a cycle, and $[\zeta] = [\zeta']$. Applying g_* we see that

$$g_*(\zeta') = g_*(\zeta) - g_*(\partial b_{n+1}) = g_*(\zeta) - \partial(f_*(b_{n+1})) = g_*(\zeta) - g_*(\zeta) = 0$$

as a chain, so exactness implies there exists an $a_n \in A_n$ so that $f_*(a_n) = \zeta'$. Now, we need to show $\partial(a_n) = 0$. We have $\partial\zeta' = 0$ and therefore, by commutativity, $f_*(\partial(a_n)) = 0$, but by exactness at A_{n-1}, f_* is an injection, and so $\partial(a_n) = 0$. Thus, $f_*[a_n] = [\zeta'] = [\zeta]$.

Construction of the connecting homomorphism, β, makes use of the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A_{n-2} & \xrightarrow{f_{n-1}} & B_{n-2} & & & &
\end{array}$$

Start with a homology class $[c] \in H_n(C_*)$ represented by a cycle $c \in C_n$. By exactness at C_n there is a $b_n \in B_n$ so that $g_n(b_n) = c$. By comutativity of the diagram $g_{n-1}(\partial b_n) = \partial g_n(b_n) = \partial c = 0$, since c is a cycle. Then by exactness at B_{n-1}, since $b_{n-1} \in \text{Ker}(g_{n-1}) = \text{Im}(f_{n-1})$ there is an element $a_{n-1} \in A_{n-1}$ so that $f_{n-1}(a_{n-1}) = \partial b_n$. We need to show that a_{n-1} is a cycle. By commutativity of the diagram $f_{n-2}(\partial a_{n-1}) = \partial f_n(a_{n-1}) = \partial\partial b_{n-1} = 0$. Then by exactness at A_{n-1}, f_{n-1} is an injection, so $\partial a_{n-1} = 0$. Thus, a_{n-1} is a cycle.

We made a choice when we picked a b_n in the pre-image of c. If we choose another element b'_n so that $g_n(b'_n) = c$, we have $g_n(b'_n - b_n) = g_n(b'_n) - g_n(b_n) = c - c = 0$. That is, b_n and b'_n differ by an element in the Kerg_n =Imf_n. So we have $b'_n = b_n + f_n(a_n)$ for some $a_n \in A_n$. Following this through our construction, we see that this changes a_{n-1} to $a_{n-1} + \partial a_n$, and thus does not change our map on homology.

Now, suppose we choose a different cycle representative for $[c]$, say $c + \partial c'_{n+1}$ for some $c'_{n+1} \in C_{n+1}$. Then, by exactness at C_{n+1} there is an element $b'_{n+1} \in B_{n+1}$ so that $g_{n+1}(b'_{n=1}) = c'_{n+1}$. Then a natural choice for an element in B_n that maps to $c + \partial c'_{n+1}$ under g_n is $b + \partial b'_{n+1}$. Notice we can choose any element in the pre-image of $c + \partial c'_{n+1}$ that we want, since we have already shown that our construction is independent of this choice. But now, if we continue with our construction, we see that we get the same element $a_{n-1} \in A_{n-1}$ as before. So, we have a well defined map $\beta : H_n(C_*) \to H_{n-1}(A_*)$ given by $[c] \mapsto [a_{n-1}]$. We leave the proof of exactness at $H_*(A_*)$ and $H_*(C_*)$ as an exercise. $\square$

Now that we have proven this lemma, we need to find an appropriate short exact sequence of chain complexes that will give rise to the Mayer-Vietoris sequence in homology. At first glance, the most likely candidate would be:

$$0 \longrightarrow S_*(U \cap V) \xrightarrow{(i_U)_* - (i_V)_*} S_*(U) \oplus S_*(V) \xrightarrow{(j_U)_* + (j_V)_*} S_*(X) \longrightarrow 0$$

If this was a short exact sequence then we would be done with the proof. The problem with this choice is that the sequnce is not exact at $S_*(X)$. There are singular chains in

X that do not come from the inclusions of singular chains in U and singular chains in V, those chains that have maps $\sigma : \Delta^n \to X$ with neither $\operatorname{Im}(\sigma) \nsubseteq U$ nor $\operatorname{Im}(\sigma) \nsubseteq V$. To force exactness at this point we replace $S_*(X)$ with the subcomplex $S_*^{\text{small}}(X)$ generated by the singular simplices whose images lie in either U or V. We call these chains, and the simplices of which they are made *small*. Then one can easily check that

$$0 \longrightarrow S_*(U \cap V) \xrightarrow{(i_U)_* - (i_V)_*} S_*(U) \oplus S_*(V) \xrightarrow{(j_U)_* + (j_V)_*} S_*^{\text{small}}(X) \longrightarrow 0$$

is a short exact sequence of chain complexes. By our homolgical lemma, this induces a long exact sequence in homology:

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(S_*^{\text{small}}(X)) \to H_{n-1}(U \cap V) \to \cdots$$

Now, we need to show that in fact, the inclusion $S_*^{\text{small}}(X) \hookrightarrow S_*(X)$ induces an isomorphism on homology. To do this we will define the subdivision map $\operatorname{sd}\colon S_*(X) \to S_*(X)$ and a chain homotopy from this map to the identity, showing that it induces the identity on homology.

First we will define a map sd, which given an n-simplex Δ^n, assigns a chain in $S_n(\Delta^n)$. In order to define this map it will be convinient to think of the n-simplex in three equivalent ways. We can write the n-simplex as a sequence of inclusions of faces $\Delta^n = \tau^0 \subset \tau^1 \subset \cdots \subset \tau^n$, where τ^i is an i-face of Δ^n. There are exactly $(n+1)!$ ways of doing this corresponding to the $(n+1)!$ elements of the permutation group Σ_{n+1} of the $n+1$ vertices of Δ^n. To see this, we give a correspondance between ordered lists of the $n+1$ vertices, and the expression of Δ^n as a sequence of inclusions of faces. Given an ordered list of the vertices $(v_0, v_1, \ldots, v_n)$ we let τ^i be the face of Δ^n with vertices $v_0, v_1, \ldots, v_i$. Given a sequence of faces, $\Delta^n = \tau^0 \subset \tau^1 \subset \cdots \subset \tau^n$, we obtain an ordered list of vertices by letting $v_0 = \tau_0$, v_1 the vertex of τ_1 not in τ_0 and so on. Then given a permutation $p \in \Sigma_{n+1}$, we have the sequence of faces corresponding to the ordered list of vertices $(v_{p(0)}, v_{p(1)}, \ldots, v_{p(n)})$. So we have the following three ways of thinking of the n-simplex:

$$\Sigma_{n+1} \Leftrightarrow \{\text{ordered lists of vertices}\} \Leftrightarrow \{\text{sequences of inclusions of faces}\}$$

For a face τ^i of an n-simplex define $\hat{\tau}^i$ to be the image under τ^i of $(1/(i+1), \ldots, 1/(i+1))$. The point $\hat{\tau}^i$ is called the *barycenter* of τ^i. Then, given a sequence of inclusions of faces $\tau^0 \subset \tau^1 \subset \cdots \tau^n$, we obtain a map of the n-simplex into itself given by the inclusion of the n-simplex with vertices the barycenters of the faces:

$$\tau^0 \subset \tau^1 \subset \cdots \subset \tau^n \rightsquigarrow (\hat{\tau}^0, \ldots, \hat{\tau}^n) : \Delta^n \hookrightarrow \Delta^n$$

Now we define $\operatorname{sd}(\Delta^n) \in S_n(\Delta^n)$ by

$$\operatorname{sd}(\Delta^n) = \sum_{\tau^0 \subset \tau^1 \subset \cdots \subset \tau^n} (-1)^{|p|} (\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^n)$$

where p is the permutation corresponding to $\tau^0 \subset \tau^1 \subset \cdots \subset \tau^n$ and $|p|$ is the sign of the permutation.

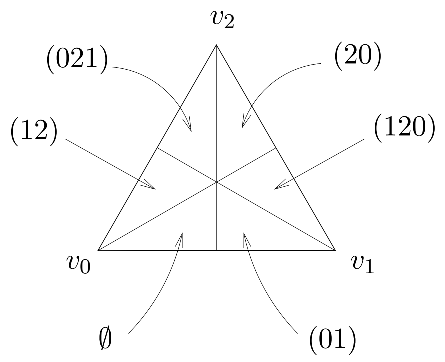

Figure 9: $sd(\Delta^2)$

Now, for $\zeta \in S_n(X)$ where $\zeta = \sum a_\sigma(\sigma : \Delta^n \to X)$ we define $\mathrm{sd}(\zeta) = \sum a_\sigma \sigma_*(\mathrm{sd}(\Delta^n)) \in S_n(X)$. We need to show that sd: $S_*(X) \to S_*(X)$ is a chain complex morphism. By naturality, it is sufficent to prove the following lemma.

Lemma 2.2.3. $\partial \mathrm{sd}(\Delta^n) = \mathrm{sd}(\partial \Delta^n)$

Proof. We have,

$$\partial sd(\Delta^n) = \partial\Bigg[\sum_{\tau^0 \subset \cdots \subset \tau^n} (-1)^{|p|}(\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^n)\Bigg] = \sum_{\tau^0 \subset \cdots \subset \tau^n} (-1)^{|p|}\partial(\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^n) =$$

$$\sum_{\tau^0 \subset \cdots \subset \tau^n} (-1)^p \Bigg[\sum_{i=0}^{n} (-1)^i (\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^{i-1}, \hat{\tau}^{i+1}, \ldots, \hat{\tau}^n)\Bigg]$$

Suppose $i \neq n$, and $\tau^0 \subset \cdots \subset \tau^{i-1} \subset \tau^{i+1} \subset \ldots \subset \tau^n$ is a chain. Then, in passing from τ^{i-1} to τ^{i+1} we add two vertices w, w'. Hence, there are exactly two $i-$simplices that we can insert at this point in the chain to form a complete chain, the corresponding permutations differ by a two-cycle interchanging w and w', and hence have opposite signs. So, all of the terms with $i < n$ cancel in pairs and we are left with

$$\partial \mathrm{sd}(\Delta^n) = \sum_{\tau^0 \subset \cdots \subset \tau^n} (-1)^{|p|}(-1)^n(\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^{n-1})$$

We can write this as,

$$\sum_{\tau^0 \subset \cdots \subset \tau^n} (-1)^{|p|}(-1)^n(\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^{n-1}) = \sum_{j=0}^{n} \Bigg[\sum_{\{\text{permutations p}|\ p(n)=j\}} (-1)^{|p|}(-1)^n(\hat{\tau}^0, \hat{\tau}^1, \ldots, \hat{\tau}^{n-1})\Bigg]$$

$$= \sum_{j=0}^{n} (-1)^j \mathrm{sd}(f_j) = \mathrm{sd}(\partial \Delta^n)$$

□

So we have a morphism of chain complexes sd: $S_*(X) \to S_*(X)$, which induces a map $\mathrm{sd}_* : H_*(X) \to H_*(X)$. Now we want to show that this induced map on homology is the identity.

Proposition 2.2.4. *The chain map* sd: $S_*(X) \to S_*(X)$ *is chain homotopic to the identity.*

Proof. We need to define $H : S_n(X) \to S_{n+1}(X)$ satisfying $\partial H + H\partial = \mathrm{sd} - \mathrm{Id}$. We will construct $H(\Delta^n) \in S_{n+1}(\Delta^n)$ by induction on n, and then define $H(\sum a_\sigma \sigma) = \sum a_\sigma \sigma_*(H(\Delta^n))$. For the initial case $n = 0$, we have $\mathrm{sd}(\Delta^0) = \Delta^0$ and $H(\Delta^0) = 0$. Suppose that $n > 0$ and that for all $k < n$ we have $H(\Delta^k)$ defined with $\partial H(\Delta^k) = \mathrm{sd}(\Delta^k) - \Delta^k - H(\partial\Delta^k)$. Consider $\mathrm{sd}(\Delta^n) - \Delta^n - H(\partial\Delta^n) \in S_n(\Delta^n)$. We want to show that this is a cycle. We have,

$$\partial\Big(\mathrm{sd}(\Delta^n) - \Delta^n - H(\partial\Delta^n)\Big) = \mathrm{sd}(\partial\Delta^n) - \partial\Delta^n - \partial H(\partial\Delta^n).$$

The inductive hypothesis gives

$$\partial H(\partial\Delta^n) = \mathrm{sd}(\partial\Delta^n) - \partial\Delta^n - H(\partial\partial\Delta^n) = \mathrm{sd}(\partial\Delta^n) - \partial\Delta^n$$

Thus,

$$\partial\Big(\mathrm{sd}(\Delta^n) - \Delta^n - H(\partial\Delta^n)\Big) = \mathrm{sd}(\partial\Delta^n) - \partial\Delta^n - \mathrm{sd}(\partial\Delta^n) + \partial\Delta^n = 0.$$

Using the fact that $H_n(\Delta^n) = 0$, we see that this cycle is also a boundary. Choose $H(\Delta^n) \in S_{n+1}(\Delta^n)$ to be some element so that

$$\partial H(\Delta^n) = \mathrm{sd}(\Delta^n) - \Delta^n - H(\partial\Delta^n).$$

□

The next step of the proof will be to show that by repeatedly applying the sd map to elements of $S_*(X)$ we can map all of $S_*(X)$ into $S_*^{\mathrm{small}}(X)$. We make use of the following lemma.

Lemma 2.2.5. *The diameter of every simplex in sd*$(\Delta^n) \leq \frac{n}{n+1}$ *diam*(Δ^n).

Proof. The details of the proof are left to the reader. The general idea is to first reduce to showing that the distance from any vertex v_i of $\Delta^n = [v_0, \dots, v_n]$ to the barycenter b is less than or equal to $\frac{n}{n+1}$ *diam*(Δ^n). Then, let b_i be the barycenter of the face $f_i = [v_0, \dots, \hat{v_1}, \dots, v_n]$ of Δ^n. Then, $b = \frac{1}{n+1}v_i + \frac{n}{n+1}b_i$. Notice that this implies b lies on the line segment between v_i and b_i, and the distance from v_i to b is $n/n+1$ times the length of $[v_i, b_i]$. This shows that the distance from b to v_i is bounded by $n/n+1$ times the diameter of $[v_0, \dots, v_n]$. □

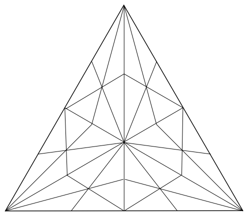

Figure 10: $sd^2(\Delta^2)$

Corollary 2.2.6. *For $\sigma : \Delta^n \to X$ there exists a constant k such that $\mathrm{sd}^k(\sigma)$ is small.*

Proof. Let $W = \sigma^{-1}(U)$ and $Y = \sigma^{-1}(V)$. Then W and Y form an open cover of Δ^n. Since Δ^n is compact, there is some number λ such that for any $x \in Delta^n$ either $B_\lambda(x) \subset W$ or $B_\lambda(x) \subset Y$ (This is a standard argument from point-set topology, look under Lebesgue number in any basic topology text such as *Munkres*). Then there is some k such that $(\frac{n}{n+1})^k diam(\Delta^n) < \lambda$, and thus $\mathrm{sd}^k(\sigma)$ is small. □

Corollary 2.2.7. *For any n-chain $\zeta \in S_n(X)$ there exists a k so that $sd^k(\zeta) \in S_n^{\text{small}}(X)$.*

Corollary 2.2.8. *The map on homology induced by $S_*^{\text{small}}(X) \hookrightarrow S_*(X)$ is onto.*

Proof. Given $[\zeta] \in H_n(X)$, let ζ be a representative cycle for this homology class. Then $\mathrm{sd}^k(\zeta)$ is also a cycle, and $\mathrm{sd}^k(\zeta)$ is homologus to ζ, but if k is sufficently large $\mathrm{sd}^k(\zeta)$ is in $S_n^{\text{small}}(X)$. □

Remark 2.2.9. If ζ is small, then $\mathrm{sd}(\zeta)$ and $H(\zeta)$ are also small.

Proposition 2.2.10. *The map on homology induced by $S_*^{\text{small}}(X) \hookrightarrow S_*(X)$ is injective.*

Proof. Suppose $[a'] \in H_n^{\text{small}}(X)$, and $a' = \partial b$ where $b \in S_{n+1}(X)$ i.e. $[a'] = 0 \in H_n(X)$. We need to show that $[a'] = 0 \in H_n^{\text{small}}(X)$, that is $a' = \partial b'$, for some $b' \in S_{n+1}^{\text{small}}(X)$. By corollary 2.2.7 there is a k such that $\mathrm{sd}^k(b) = b' \in S_{n+1}^{\text{small}}(X)$. Then subdivide a' k-times,

$$\mathrm{sd}^k(a') = \mathrm{sd}^k(\partial b) = \partial(\mathrm{sd}^k(b)) = \partial(b').$$

Thus, $[\mathrm{sd}^k(a')] = 0$ in $H_n^{\text{small}}(X)$. Lastly, we need to show $[a'] = [\mathrm{sd}^k(a')] \in H_n^{\text{small}},$. This is proved by induction on k using remark 2.2.9 □

This completes the proof of Mayer-Vietoris.

As an application of this result and the computation for contractible spaces (which is a special case of the homotopy axiom), let us compute the homology groups of the spheres.

Corollary 2.2.11. *For $n \geq 1$, the homology $H_k(S^n)$ is zero unless $k = 0, n$. For these two values of k the homology is isomorphic to $\mathbb{Z}$.*

Proof. We proceed by induction on n. Suppose that we have

$$H_k(S^{n-1}) = \begin{cases} \mathbb{Z} & k = n-1, 0 \\ 0 & \text{otherwise} \end{cases}$$

for some $n - 1 \geq 1$. Choose a point $p \in S^n$ and let p^* be the antipodal point. Let $U = S^n - \{p\}$ and $V = S^n - \{p^*\}$. Then $\{U, V\}$ is an open cover of S^n. Applying Mayer-Vietoris, we obtain the long exact sequence:

$$\cdots \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(S^n) \to H_{k-1}(U \cap V) \to \cdots.$$

Both U and V are homeomorphic to $\mathbb{R}^n$ and hence are contractible. Then by the homotoppy axiom,

$$H_*(U) = H_*(V) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

As an exercise, show that, $U \cap V = S^n - \{p\} - \{p^*\}$ is homotopy equivalent to S^{n-1}, and so by the homotopy axiom and the inductive hypothesis,

$$H_*(U \cap V) \cong H_*(S^{n-1}) \cong \begin{cases} \mathbb{Z} & * = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Putting these into the Mayer-Vietoris long exact sequence, we see that for $k \geq 2$. $H_k(S^n) \cong H_{k-1}(U \cap V)$ and so the result follows by induction, and since,

$$\begin{tikzcd}
0 \arrow[r] & H_1(S^n) \arrow[r] \arrow[d, "="'] & H_0(S^{n-1}) \arrow[r] \arrow[d, "\cong"'] & H_0(U) \oplus H_0(V) \arrow[r] \arrow[d, "\cong"'] & H_0(S^n) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & H_0(S^n) \arrow[r] & 0
\end{tikzcd}$$

We still need to show the intitial step. We have already shown that the homology of S^1 is isomorphic to $\mathbb{Z}$ in degree 1 by direct computation. We also know that $H_0(S^1) \cong \mathbb{Z}$ since S^1 has a single path component. We need only to verify that $H_k(S^1) = 0$ for $k > 1$. If we let $\{U, V\}$ be an open cover of S^1 as above for the higher dimensional spheres, then again, both U and V deformation retract to a point and $U \cap V$ is homotopic to S^0, which is just two points. Applying Mayer-Vietoris to S^1 with this open cover shows, $H_k(S^1) = 0$ for all $k > 1$. $\square$

As another application, we have the extension of the Brouwer fixed point theorem to higher dimensional disks.

Theorem 2.2.12. *Let $D^n \subset \mathbb{R}^n$ be the closed unit disk, any map $D^n \to D^n$ has a fixed point.*

Proof. The proof is left as an exercise. It is the same as that for the lower dimensional case, now using the fact that $H_{n-1}(S^{n-1}) \neq 0$ whereas $H_{n-1}(D^n) = 0$. □

Exercise 2.2.13. *Show that if $X = U \coprod V$, then $H_*(X) = H_*(U) \oplus H_*(V)$*

Exercise 2.2.14. *Prove theorem 2.2.12, the extension of the Brouwer Fixed Point Theorem to higher dimensional disks.*

Exercise 2.2.15. *Suppose that $x \in X$ and $y \in Y$ have open neighborhoods that strongly deformation retract to x and to y respectively, i.e. there are open neighborhhods $U \subset X$ and $V \subset Y$ and maps $H_U : U \times I \to U$ and $H_V : V \times I \to V$ with $H_U(u, 0) = u$, $H_U(u, 1) = x$ and $H_U(x, t) = x$ for all $0 \leq t \leq 1$, and $H_V(v, 0) = v$, $H_V(v, 1) = x$ and $H_V(y, t) = y$ for all $0 \leq t \leq 1$. Let $X \wedge Y$ be the one point union of X and Y where x is idenified with y. Show that $H_n(x \wedge Y) = H_n(X) \oplus H_n(Y)$ for all $n > 0$.*

Exercise 2.2.16. *Let $Z \subset X$ and $Z \subset Y$ have open neighborhhods that strong deformation retract to them. Let $W = X \cup_Z Y$. Show that there is a long exact sequence*

$$\cdots \longrightarrow H_k(Z) \longrightarrow H_k(X) \oplus H_k(Y) \longrightarrow H_k(W) \longrightarrow H_{k-1}(Z) \longrightarrow \cdots$$

2.3 Relative Homology and the Long Exact Sequence of a Pair

Definition 2.3.1. A pair of topological spaces (X, A) consists of a topological space X and a subspace A of X. These objects form a category whose morphisms $(X, A) \to (Y, B)$ are continuous maps from X to Y mapping A to B.

If (X, A) is a pair of topological spaces then $S_*(A)$ is naturally a subcomplex of $S_*(X)$. In fact, for each k there is the natural basis for $S_k(X)$ and the group $S_k(A)$ is the subgroup generated by the subset of singular simplices whose image is contained in A. We define $S_*(X, A)$ to be the quotient chain complex $S_*(X)/S_*(A)$. It is a chain complex whose chain groups are free abelian groups. We define the relative homology $H_*(X, A)$ to be the homology of $S_*(X, A)$. Notice that a topological space X can be identified with an object of the category of pairs namely $(X, \emptyset)$. We have the inclusion $(X, \emptyset) \subset (X, A)$.

Relative homology is a functor from the category of pairs of topological spaces to the category of graded abelian groups.

For any pair of topological spaces (X, A), we have a short exact sequence of chain complexes:

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$$

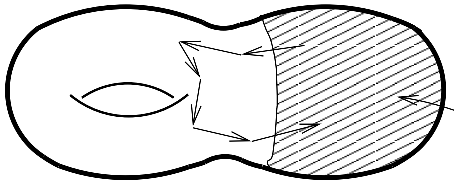

Figure 11: A relative cycle

leading to a long exact sequence of homology

$$\cdots \longrightarrow H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\beta} H_{k-1}(A) \longrightarrow \cdots$$

where $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ are the inclusions, and β is the connecting homomorphism. This long exact sequence is called the *long exact sequence of a pair*, and is functorial for morphisms in the category of pairs of topological spaces.

There is also a relative version of the Mayer-Vietoris sequence.

Proposition 2.3.2. *If (X, A) is a pair of topological spaces and $\{U, V\}$ is an open cover of X, then we have the following long exact sequence in homology,*

$$\cdots \to H_k(U\cap V, U_A\cap V_A) \to H_k(U, U_A)\oplus H_k(V, V_A) \to H_k(X, A) \to H_{k-1}((U\cap V, U_A\cap V_A) \to \cdots$$

where $U_A = U \cap A$ and $V_A = V \cap A$.

Proof. The proof is essentially the same as the proof of Mayer-Vietoris in the absolute case. □

2.4 The Excision Axiom for Singular Homology

Our next axiom tells us that we can cut out, or *excise*, subspaces from topological pairs without affecting the relative homology, given a few small assumptions about the subspaces.

Theorem 2.4.1. (Excision) *Let (X, A) be a pair of topological spaces and let $K \subset A$ be such that $\overline{K} \subset \mathrm{int}(A)$. Then the natural inclusion $(X \setminus K, A \setminus K)$ to (X, A) induces an isomorphism on homology.*

Proof. The proof uses the tools developed in the proof of Mayer-Vietoris. Let $B = X \setminus K$. Then $A \cap B = A \setminus K$, so we need to show the inclusion induces $H_*(B, A \cap B) \cong H_*(X, A)$. Notice that the interiors of A and B form an open cover of X. Thus, using the techniques of the Mayer-Vietoris proof, we can subdivide a chain in $S_*(X)$ to a chain which is small with respect to the open cover $\{\mathrm{int}A, \mathrm{int}B\}$, and the inclusion $S_*^{small}(X) \hookrightarrow S_*(X)$ induces an isomporphism in homology. Recall that $S_*^{small}(X)$ is generated by singular simplices whose images lie in either int$A$ or int$B$. In particular, the images of the generating simplices lie in either A or B since int$A \subset A$ and int$B \subset B$. For our purposes, this fact will be more useful, so let $S_*^{A+B}(X)$ denote the singular chains generated by singular simplices whose images lie in either A or B.

Lemma 2.4.2. *The inclusion $S_*^{A+B}(X) \hookrightarrow S_*(X)$ induces an isomorphism in homology.*

Proof. The proof is almost exactly the same as in the Mayer-Vietoris proof. First we show that the induced map on homology is onto. Given $[\zeta] \in H_n(X)$, let ζ be a representative cycle for this homology class. Then $\mathrm{sd}^k(\zeta)$ is also a cycle, and $\mathrm{sd}^k(\zeta)$ is homologus to ζ, but if k is sufficently large $\mathrm{sd}^k(\zeta)$ is in $S_n^{\mathrm{small}}(X) \subset S_n^{A+B}(X)$. Now we show injectivity. Suppose $[a'] \in H_n^{A+B}(X)$, and $a' = \partial b$ where $b \in S_{n+1}(X)$ i.e. $[a'] = 0 \in H_n(X)$. We need to show that $[a'] = 0 \in H_n^{A+B}(X)$, that is $a' = \partial b'$, for some $b' \in S_{n+1}^{A+B}(X)$. By corollary 2.2.7 there is a k such that $\mathrm{sd}^k(b) = b' \in S_{n+1}^{\mathrm{small}}(X) \subset S_{n+1}^{A+B}(X)$. Then subdivide k-times giving,

$$\mathrm{sd}^k(a') = \mathrm{sd}^k(\partial b) = \partial(\mathrm{sd}^k(b)) = \partial(b').$$

Thus, $[\mathrm{sd}^k(a')] = 0$ in $H_n^{A+B}(X)$. Lastly, we need to show $[a'] = [\mathrm{sd}^k(a')] \in H_n^{A+B}$, but this is true since if we start with chains in $S_*^{A+B}(X)$, and subdivide or apply our chain homotopy, we still have chains in $S_*^{A+B}(X)$. □

Furthermore, if we start with a chain in A, subdividing it still gives a chain in A, so if we quotient out by chains in A, the map $S_*^{A+B}(X)/S_*(A) \to S_*(X)/S_*(A)$ still induces an isomorphism in homology. The map $S_*(B)/S_*(A \cap B) \to S_*^{A+B}(X)/S_*(A)$ induced by inclusion is an isomporphism on the chain level, since both groups are generated by singular simplices whose images are in B but not in A. Putting these together we obtain the desired isomorphism in homology. □

2.5 The Dimension Axiom

There is one more axiom, which we have already proven, the *dimension axiom*.

Theorem 2.5.1. Dimension Axiom *If X is a point then,*

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. See proposition 1.3.8. □

2.6 Reduced Homology

For any non-empty space X we denote by $\tilde{H}_*(X)$, the *reduced singular homology* of X. By definition it is the kernel of the natural mapping $H_*(X) \to H_*(p)$ where p is the one-point space. Thus, $H_k(X) = \tilde{H}_k(X)$ for all $k > 0$ and $\tilde{H}_0(X)$ is trivial if X is path connected. The long exact sequence of the pair remains valid if we replace the singular homology of X and A by their reduced singular homology but leave the homology of the pair unchanged. Because of the removal of the trivial $\mathbb{Z}$ in degree zero, reduced homology is often cleaner to work with. As an example, we compute the relative homology of the disk modulo its boundary using the reduced homology.

Proposition 2.6.1. *$H_k(D^n, S^{n-1}) = 0$ unless $k = n$, in which case the relative homology is isomorphic to $\mathbb{Z}$.*

Proof. The long exact sequence of a pair gives,

$$\cdots \longrightarrow \tilde{H}_k(D^n) \longrightarrow H_k(D^n, S^{n-1}) \longrightarrow \tilde{H}_{k-1}(S^{n-1}) \longrightarrow \tilde{H}_{k-1}(D^n) \longrightarrow \cdots$$

But, since D^n is contractible, for $k > 0$ we have,

$$0 \longrightarrow H_k(D^n, S^{n-1}) \xrightarrow{\cong} H_{k-1}(S^{n-1}) \longrightarrow 0$$

The $k = 0$ case is clear. Thus,

$$H_k(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases}$$

□

3 Applications of Singular Homology

Now we give some of the nicest applications of the computations of the homology of the disks and spheres.

3.1 Invariance of Domain

Corollary 3.1.1. (Invariance of Domain) *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be non-empty open subsets. If U and V are homeomorphic, then $n = m$.*

Proof. We begin with a lemma,

Lemma 3.1.2. *Let U be a non-empty open subset of $\mathbb{R}^n$ and let $x \in U$. Then $H_k(U, U \backslash \{x\})$ is zero except when $k = n$ in which case the relative homology group is $\mathbb{Z}$.*

Proof. Let $U \subset \mathbb{R}^n$ be a non-empty open set. Let $x \in U$. If we let $K = \mathbb{R}^n \setminus U$, then K is closed and $K \subset \mathbb{R}^n \setminus \{x\}$. So applying excision with $X = \mathbb{R}^n$, $A = \mathbb{R}^n \setminus \{x\}$ and $K = \mathbb{R}^n \setminus U$, we have,

$$H_*(U, U \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

Since $\mathbb{R}^n$ is contractible, we have $\tilde{H}_*(\mathbb{R}^n) = 0$, and by the homotopy axiom $\tilde{H}_*(\mathbb{R}^n \setminus \{x\}) = \tilde{H}_*(\mathbb{R}^n \setminus \{0\}) = \tilde{H}_*(S^{n-1}) = \mathbb{Z}$ if $* = n-1$ and 0 otherwise. And now, using the long exact sequence of a pair we have,

$$\tilde{H}_*(\mathbb{R}^n) = 0 \to H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \tilde{H}_{*-1}(S^{n-1}) \to \tilde{H}_{*-1}(\mathbb{R}^n) = 0,$$

and thus $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{*-1}(S^{n-1}) = \mathbb{Z}$ if $* = n$ and 0 otherwise. So, $H_*(U, U \setminus \{x\}$ is non-zero in exactly one dimension, the dimension of the euclidean space in which U sits as an open set, and in that dimension it is isomorphic to $\mathbb{Z}$. □

Now, suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic non-empty open sets, with $\phi : U \to V$ a homeomorphism. Let $x \in U$ and $y = \phi(x) \in V$. Then ϕ induces a homeomorphism of pairs (i.e. an isomorphism in the category of pairs of topological spaces)

$$\phi : (U, U \setminus \{x\}) \to (V, V \setminus \{y\}),$$

and thus the induced map on homology is also an isomorphism,

$$\phi_* : H_*(U, U \setminus \{x\}) \to H_*(V, V \setminus \{y\}).$$

Thus, by lemma 3.1.2, $n = m$. □

3.2 The Jordan Curve Theorem and its Generalizations

The Jordan curve theorem says that any simple closed curve in the plane divides the plane into two pieces, and is the frontier of each piece. Here is a homological theorem which applies to all dimensions and as we shall see easily implies the classical Jordan curve theorem.

Theorem 3.2.1. *For $k < n$, if $\psi : S^k \hookrightarrow S^n$ is a homeomorphism onto its image then $\tilde{H}_i(S^n - \psi(S^k)) = 0$ unless $i = n - k - 1$ in which case the homology group is isomorphic to $\mathbb{Z}$.*

Proof. First, we will prove the following lemma.

Lemma 3.2.2. *If I^k is a cube $(k \leq n)$ and $\phi : I^k \hookrightarrow S^n$ is a continuous one-to-one mapping. Then $\tilde{H}_*(S^n - \phi(I^k)) = 0$.*

Proof. The proof is by induction on k. For $k = 0$, we know that $S^n - \{pt\} \cong \mathbb{R}^n$, and $\tilde{H}_*(\mathbb{R}^n) = 0$. Now, suppose the result holds for $k - 1$. Let

$$\begin{aligned} Y_- &= \phi(I^{k-1} \times [0, 1/2]) \\ Y_+ &= \phi(I^{k-1} \times [1/2, 1]) \\ Y_{1/2} &= \phi(I^{k-1} \times [1/2]) \end{aligned}$$

Then,

$$\begin{aligned} (S^n - Y_+) \cup (S^n - Y_-) &= S^n - Y_{1/2} \\ (S^n - Y_+) \cap (S^n - Y_-) &= S^n - Y. \end{aligned}$$

Then, by Mayer-Vietoris we have the following exact sequence,

$$\tilde{H}_*(S^n - Y_{1/2}) \to \tilde{H}_*(S^n - Y) \to \tilde{H}_*(S^n - Y_-) \otimes \tilde{H}_*(S^n - Y_+) \to \tilde{H}_{*-1}(S^n - Y_{1/2})$$

Since, $\tilde{H}_{*-1}(S^n - Y_{1/2}) = 0$, this implies,

$$\tilde{H}_*(S^n - Y) \cong \tilde{H}_*(S^n - Y_-) \otimes \tilde{H}_*(S^n - Y_+).$$

Now, if $a \in H_i(S^N - Y)$ and $a \neq 0$, then either $(i_-)_*(a) \in H_i(S^n - Y_-) \neq 0$ or $(i_+)_*(a) \in H_i(S^n - Y_+) \neq 0$, where $i_- : S^n - Y \hookrightarrow S^n - Y_-$ and $i_+ : S^n - Y \hookrightarrow S^n - Y_+$ are the inclusions. Choose $Y_1 \subset Y$ so that the inclusion of a is non-zero in $H_i(S^n - Y_1)$. Repeat this step to get $Y_2, Y_3, \ldots$ so,

$$Y_m = \phi\left(I^{k-1} \times \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]\right)$$

and, $a_m \neq 0 \in H_i(S^n - Y_m)$. Then let $Y_\infty = \bigcap_{m=1}^{\infty} Y_m = \phi(I^{k-1} \times \{x\})$. Then $a \mapsto 0 \in H_i(S^n - Y_\infty)$. Replace a by a cycle ζ in $S^n - Y$, where $\zeta = \partial c$ for some c in $S^n - Y_\infty$. That is $c = \sum n_\sigma \sigma$ where $\bigcup_{\sigma,\ n_\sigma \neq 0} \mathrm{Im}(\sigma) \subset S^n - Y_\infty$. Now, there exists an open set $V \subset S^n$ so that $Y_\infty \subset V$ and $\bigcup_{\sigma,\ n_\sigma \neq 0} \mathrm{Im}(\sigma) \subset V$, which implies $Y_m \subset V$ for m sufficently large. Thus, $c \in S_{i+1}(S^n - Y_m)$ for all $m \gg 1$, and hence, $a_m = 0$ for all $m \gg 1$. This is a contradiction. □

Now we prove the theorem by induction on k. For $k = -1$, $S^{-1} = \emptyset$ and

$$\tilde{H}_*(S^n - \psi(S^k)) = \tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & otherwise \end{cases}$$

Now, suppose the result holds for k, we want to show that it holds for $k+1$. We have $S^{k+1} = D_+^{k+1} \cup D_-^{k+1}$, where $D_\pm$ are disks homeomorphic to I^{k+1}. Now, consider $\psi : S^{k+1} \hookrightarrow S^n$.

Let $U = S^n - \psi(D_+^{k+1})$ and $V = S^n - D_-^{k+1}$. Then $U \cup V = S^n - \psi(S^k)$, call this X, and $U \cap V = S^n - \psi(S^{k+1})$. Now, U and V are both open in S^n, and hence are both open in X. By lemma 3.2.2 $\tilde{H}(U) = \tilde{H}(V) = 0$, so the Mayer-Vietoris sequence,

$$\tilde{H}_*(U) \otimes \tilde{H}_*(V) \to \tilde{H}_*(S^n - \psi(S^k)) \to \tilde{H}_{*-1}(S^n - \psi(S^{k+1})) \to 0,$$

gives an isomorphism,

$$\tilde{H}_*(S^n - \psi(S^k)) \cong \tilde{H}_{*-1}(S^n - \psi(S^{k+1})),$$

and the result follows by induction. □

Corollary 3.2.3. *Any embedding ψ of S^{n-1} into S^n separates S^n into exactly two components. Furthermore, if $S^n - \psi(S^{n-1}) = X \amalg Y$ with X and Y open, then $\psi(S^{n-1}) = \overline{X} \cap \overline{Y}$.*

Proof. By the theorem, $\tilde{H}_0(S^n - \psi(S^{n-1})) \cong \mathbb{Z}$ and hence, $H_0(S^n - \psi(S^{n-1})) \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore $S^n - \psi(S^{n-1})$ has two path components, but since S^n is locally path connected, so is the open subset $S^n - \psi(S^{n-1})$ and thus path components are components.

Now, suppose $S^n - \psi(S^{n-1}) = X \amalg Y$ with X and Y open, then $\bar{X} \subset S^n - Y$ since $X \subset S^n - Y$ and Y is open. Similarly, $\bar{Y} \subset S^n - X$. So, $X \subset \bar{X} \subset [X \cup \psi(S^{n-1})]$ and $Y \subset \bar{Y} \subset [Y \cup \psi(S^{n-1})]$. We will show that $\psi(S^{n-1}) \subset \bar{X}$ and $\psi(S^{n-1}) \subset \bar{Y}$. Suppose not. By symmetry we may assume that $\psi(S^{n-1}) \not\subseteq \bar{X}$. Then there exists a point $p \in \psi(S^{n-1})$ such that $p \notin \bar{X}$. Now, $p \in \psi(S^{n-1}) - \bar{X}$ and therefore, for some $\epsilon > 0$, $\psi(D_\epsilon(p)) \subset \psi(S^{n-1}) - \bar{X}$. Let $Z = X \cup \psi(D_\epsilon^{n-1}(p)) \cup Y$. Then $\bar{X} \cap Z = X = X \cap Z$, and so Z is not connected, but $Z = S^n - \psi(S^{n-1} - D_\epsilon(p))$ and $S^{n-1} - D_\epsilon(p) \cong D^{n-1}$, so $\tilde{H}_0(Z) = 0$, by our lemma above. Contradiction. □

In particular, this result is true for any simple closed curve in the plane. This result is known as the Jordan curve theorem.

Corollary 3.2.4. (The Jordan Curve Theorem) *Suppose $C \subset \mathbb{R}^2$ is a simple closed curve i.e. C is homeomporphic to S^1. Then $\mathbb{R}^2 \setminus C$ has exactly two components and C is the frontier of each compnent.*

Exercise 3.2.5. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. Compute the induced map $L_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.*

Exercise 3.2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with $f(0) = 0$. Compute the induced map $f_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.*

Exercise 3.2.7. *In corollary 2.2.11 we saw that for $n \geq 1$*

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \textit{otherwise} \end{cases}$$

Therefore, if $f : S^n \to S^n$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by some integer d. We call this integer the degree *of the map f. Compute the degree of the identity map on S^n and the antipodal map $a : S^n \to S^n$.*

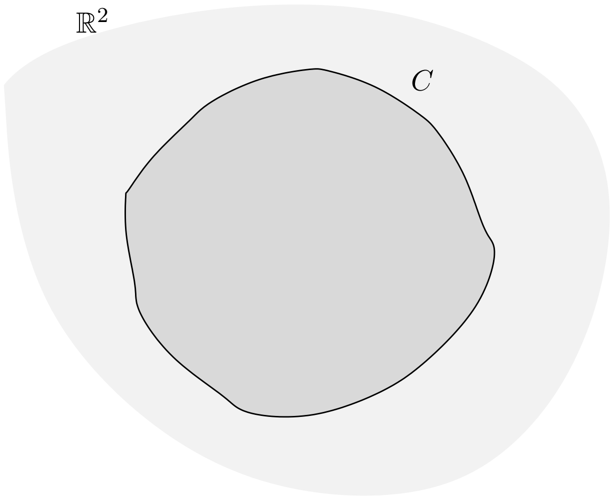

Figure 12: Two components of $\mathbb{R}^2$ separated by a simple closed curve C

Exercise 3.2.8. *For n even show that every map $S^n \to S^n$ homotopic to the identity has a fixed point. Show that this is not true for every n odd, by showing that the idenitity is homotopic to the antipodal map for n odd.*

Exercise 3.2.9. *Compute $H_*(S^n \times X$ in terms of $H_*(X)$.*

Exercise 3.2.10. *Let $X = S^p \vee S^q$. compute $H_*(X)$.*

3.3 Cellular (CW) Homology

Let $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^N = X$ be a finite CW complex (see appendix C). We define a chain complex with chain groups $C_k^{cw}(X) = H_k(X^k, X^{k-1})$ and boundary map $\partial_{cw} : C_k^{cw}(X) \to C_{k-1}^{cw}(X)$ given by the composition,

$$C_k^{cw}(X) = H_k(X^k, X^{k-1}) \xrightarrow{\partial} H_{k-1}(X^{k-1}) \xrightarrow{i_*} H_{k-1}(X^{k-1}, X^{k-2}) = C_{k-1}^{cw}(X)$$

Proposition 3.3.1. $\partial_{cw}^2 = 0$

Proof. The cellular boundary map fits into a commutative diagram involving portions of the long exact sequences of the pairs (X^{n+1}, X^n) and (X^n, X^{n-1}).

$$
\begin{tikzcd}
 & 0 & & & \\
 & H_n(X^{n+1}) \cong H_n(X) \arrow[u] & 0 \arrow[d] & & \\
 & H_n(X^n) \arrow[u] \arrow[r, equal] & H_n(X^n) \arrow[d, "i_*"'] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_{n+1}"] \arrow[r, "\partial_{cw}"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial_{cw}"] \arrow[d, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & H_{n-1}(X^{n-1}) \arrow[r, equal] & H_{n-1}(X^{n-1}) \arrow[u, "j_*"] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}
$$

If we trace through this diagram we see that the composition ∂_{cw}^2 factors through the 0 map, and is thus 0. □

So, $\{C_*^{cw}(X), \partial_{cw}\}$ forms a chain complex, and we define the cellular homology of a CW complex X to be the homology of this chain complex, $H_*^{cw}(X) = H_*(\{C_*^{cw}(X), \partial_{cw}\})$.

Lemma 3.3.2. *$H_*(X^k) = 0$ for $* > k$*

Lemma 3.3.3. *$H_*(X^k) = H_*(X)$ for $* < k$*

Claim 3.3.4. *$H_*^{cw}(X)$ is identified with the singular homology of X.*

Proof. Examining the diagram in the proof above, we see that $H_n(X) \cong H_n(X^n)/\mathrm{Im}(\partial_{n+1})$. Since i_* is an injection, $i_*(\mathrm{Im}(\partial_{n+1})) \cong \mathrm{Im}(i_* \circ \partial_{n+1}) = \mathrm{Im}(\partial_{cw})$, and $i_*(H_n(X^n)) \cong \mathrm{Im}(i_*) = \mathrm{Ker}(\partial_n)$. Since j_* is an injection, $\mathrm{Ker}(\partial_n) = \mathrm{Ker}(\partial_{cw})$. Thus, i_* induces an isomorphism $H_n(X^n)/\mathrm{Im}(\partial_{n+1}) \cong \mathrm{Ker}(\partial_{cw})/\mathrm{Im}(\partial_{cw})$.

□

4 Other Homologies and Cohomologies

4.1 Singular Cohomology

The group of *singular cochains* of degree n of X, denoted $S^n(X)$, is defined to be the dual group to $S_n(X)$, i.e., $S^n(X) = \mathrm{Hom}(S_n(X), \mathbb{Z})$. Note that $S^*(X)$ is not usually s free abelian chain complex. We then define the coboundary map $\delta\colon S^n(X) \to S^{n+1}(X)$ to be the dual to $\partial : S_{n+1}(X) \to S_n(X)$, so for $\phi \in S^n(X)$, and $\sigma : \Delta^{n+1} \to X$ we have $< \delta\phi, \sigma > = < \phi, \partial\sigma >$. This forms a cochain complex, the *singular cochain complex* of X. Its cohomology is called the *singular cohomology* of X and is denoted $H^*(X)$.

Singular cohomology is a contravariant functor from the category of topological spaces to the category of graded groups. This means that if $f\colon X \to Y$ is a continuous map then we have the induced homomorphism $f^*\colon H^*(Y) \to H^*(X)$. This association preserves identities and takes compositions to compositions (with the order reversed).

We also have the notion of singular cohomology of X with coefficients in an abelian group A, denoted $H^*(X;A)$, and defined as the cohomology of the cochain complex $\mathrm{Hom}(S_*(X), A)$. If A is a ring then these cohomology groups are modules over A.

The singular homology and singular cohomology of a space are clearly very closely related, and it is not suprising that versions of the axioms for singular homology also hold in the singular cohomology setting. The following theorem will be useful in establishing those axioms.

Theorem 4.1.1. *If C_* and D_* are free abelian chain complexes with $C_*, D_* = 0$ for $* \ll 0$, and $\phi_* : C_* \to D_*$ is a chain map inducing an isomorphism on homology then the induced map on cohomology is also an isomorphism.*

Proof. The following lemma is a special case of the *Universal Coefficent Theorem.*

Lemma 4.1.2. *Let C_* be a free abelian chain complex with $C_* = 0$ for $* \ll 0$. If $H_*(C^*) = 0$, then $H^*(C_*) = 0$.*

Proof. Define $Z_n \subset C_n$ to be the cycles $=$ Ker ∂_n, and $B_{n-1} \subset C_{n-1}$ to be the boundaries $=$ Image ∂_n. Then we have the following two short exact sequences:

$$0 \to Z_n \to C_n \to B_{n-1} \to 0 \tag{a}$$

$$0 \to B_n \to Z_n \to H_n(C_*) \to 0 \tag{b}$$

Suppose that $H_*(C_*) = 0$. Then sequence (b) becomes, $0 \to B_n \to Z_n \to 0$, and so $B_n \cong Z_n$ for all n. If the C_n are free abelian and $C_{-k} = 0$ for k sufficently small, then $C_n \cong B_{n-1} \oplus Z_n$, i.e. sequence (a) splits for all n. We proceed by induction on n, starting with n sufficently small, so that $C_* = 0$. In the initial case we have,

$$0 \to Z_0 \to C_0 \to B_{-1} = 0 \to 0,$$

Suppose that C_n splits. This implies that $Z_0 = B_0$ is projective (indeed free), and thus the following sequence splits,

$$0 \to Z_1 \to C_1 \to B_0 = 0 \to 0.$$

Thus, $C_1 \cong Z_1 \oplus B_0$. Now, suppose that we have shown $C_k \cong Z_k \oplus B_{k-1}$ for all $k \leq n-1$. Then we have,

$$0 \to Z_n \to C_n \to B_{n-1} \to 0.$$

$B_{n-1} \cong Z_{n-1}$ and Z_{n-1} is a direct summand of C_{n-1}. This implies that $B_{n-1} \cong Z_{n-1}$ is projective, so $C_n \cong Z_n \oplus B_{n-1}$ and the sequence splits. We have the following situation:

$$\begin{tikzcd}
C_{n+1} \arrow[r, equal] \arrow[d, "\partial"'] & Z_{n+1} \oplus B_n \arrow[d, "\partial"'] \\
C_n \arrow[r, equal] \arrow[d, "\partial"'] & Z_n \oplus B_{n-1} \arrow[d, "\partial"'] \\
C_{n-1} \arrow[r, equal] & Z_{n-1} \oplus B_{n-1}
\end{tikzcd}$$

where in fact the ∂ maps on the left are isomorphisms $B_n \cong Z_n$, $B_{n-1} \cong Z_{n-1}$ and so on. That is, the ∂ maps are,

$$Z_n \oplus B_{n-1} \xrightarrow{\text{projection}} B_{n-1} \underset{\cong}{\xrightarrow{\partial}} Z_{n-1} \xrightarrow{\text{inclusion}} Z_{n-1} \oplus B_{n-2}$$

Dualizing, we have,

$$\begin{tikzcd}
C^{n+1} \arrow[r, equal] & Z^*_{n+1} \oplus B^*_n \\
C^n \arrow[r, equal] \arrow[u, "\delta"] & Z^*_n \oplus B^*_{n-1} \arrow[u, "\delta"] \\
C^{n-1} \arrow[r, equal] \arrow[u, "\delta"] & Z^*_{n-1} \oplus B^*_{n-1} \arrow[u, "\delta"]
\end{tikzcd}$$

where now the δ maps on the left are isomorphisms $Z^*_{n-1} \cong B^*_{n-1}$, $Z^*_n \cong B^*_n$, and so on. We have,

$$Z^*_n \oplus B^*_{n-1} \xleftarrow{\text{inclusion}} B^*_{n-1} \underset{\cong}{\xleftarrow{\delta}} Z^*_{n-1} \xleftarrow{\text{projection}} Z^*_{n-1} \oplus B^*_{n-2}$$

Thus, $H^*(C^*) = 0$. □

Remark 4.1.3. The hypothesis that $C_* = 0$, for $* \ll 0$, is actually not neccesary since it is a general fact that a subgroup of a free abelian group is free abelian.

Now, given C_* define a chain complex $C_*[k]$ by $(C_*[k])_n = C_{n-k}$ with boundary map $\partial[k]_n = \partial_{n-k}$. Then we define another chain complex, the mapping cylindar, $(M\phi)_*$ by $(M\phi)_* = C_*[1] \oplus D_*$ with boundary map

$$\partial : (M\phi)_n = C_{n-1} \oplus D_n \to (M\phi)_{n-1} = C_{n-2} \oplus D_{n-1}$$

given by

$$\partial(c_{n-1}, d_n) = (\partial^C c_{n-1}, \partial^D d_n + (-1)^n \phi_{n-1}(c_{n-1})).$$

We can express this as a matrix:

$$\partial = \begin{pmatrix} \partial_n^C & 0 \\ (-1)^n \phi_{n-1} & \partial_n^D \end{pmatrix}$$

Then to check that $\partial^2 = 0$,

$$\partial^2 = \begin{pmatrix} \partial^C & 0 \\ (-1)^n \phi & \partial^D \end{pmatrix} \begin{pmatrix} \partial^C & 0 \\ (-1)^{n-1} \phi & \partial^D \end{pmatrix} = \begin{pmatrix} (\partial^C)^2 & 0 \\ \star & (\partial^D)^2 \end{pmatrix}$$

where,

$$\star = (-1)^n \partial^D \phi + (-1)^{n-1} \partial^C = (-1)^n [\phi \partial^C - \partial^D \phi] = 0$$

since ϕ is a chain map. Then we have the following short exact sequence of chain complexes:

$$0 \to D_* \to (M\phi)_* \to C_*[1] \to 0 \tag{c}$$

and this induces the following long exact sequence on homology:

$$\cdots \longrightarrow H_n((M\phi)_*) \longrightarrow H_n(C_*[1]) \xrightarrow{\cong} H_{n-1}(D_*) \longrightarrow H_{n-1}((M\phi)_*) \longrightarrow \cdots$$

One can then check by the construction of the connecting homomorphism, $H_n(C_*[1]) \to H_{n-1}(D_*)$ in lemma 2.2.2, that this map is in fact $(-1)^n \phi_* : H_{n-1}(C_*) \to H_{n-1}(D_*)$, and is thus an isomorphism. This implies that $H_*((M\phi)_*) = 0$. Dualizing (c), we have the short exact sequence,

$$0 \to (C_*[1])^* \to (M\phi)^* \to D^* \to 0c$$

It is not true in general that the dual of an exact sequence is exact; however, since $C_*[1]$ is free abelian, the sequence can be split, and thus the dual sequence is also short exact. Using lemma 4.1.2 above, $H^*((M\phi_*)^*) = 0$ and so we have the associated long exact sequence in cohomology,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & H^n(D^*) \arrow[r, "\delta", "\cong"'] \arrow[d, equal] & H^{n+1}((C_*[1])^*) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H^n(D^*) \arrow[r, "\phi^*", "\cong"'] & H^n(C^*) & \end{tikzcd}$$

□

Now we will prove the axioms for singular cohomology.

Theorem 4.1.4. (The Homotopy Axiom for Singular Cohomology) *If $f, g : X \to Y$ are homotopic maps then $f^* = g^* : H^*(Y) \to H^*(X)$*

Proof. Let $H_n : S_n(X) \to S_{n+1}(Y)$ be a chain homotopy from f_* to g_*. So, $\partial H + H\partial = g_* - f_*$. Dualize, and we have $H^n : S^{n+1}(Y) \to S^n(X)$ satisfying $H^*\delta + \delta H^* = g^* - f^*$. Now our homological algebra from before shows that the induced maps on cohomology are equal. $\square$

Let (X, A) be a pair of topological spaces. Dual to the short exact sequence

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$$

is the short exact sequence

$$0 \to S^*(X, A) \to S^*(X) \to S^*(A) \to 0.$$

Where $S^*(X, A)$ is defined to be the kernel of the map induced by the inclusion, $i^* : S^*(X) \to S^*(A)$. This dual sequence is exact since $S_*(X, A)$ is free abelian, and hence the first short exact sequence splits. Applying cohomology yields the long exact sequence of a pair in cohomology:

Theorem 4.1.5. (The Long Exact Sequence of a Pair for Singular Cohomology) *For a pair of topological spaces (X, A), there is a long exact sequence in cohomology:*

$$\cdots \longrightarrow H^k(X, A) \longrightarrow H^k(X) \longrightarrow H^k(A) \xrightarrow{\beta} H^{k+1}(X, A) \longrightarrow \cdots$$

where the first three maps are induced by the inclusions and β is the connecting homomorphism associated to the above short exact sequence of chain complexes. Furthermore, this long exact sequence is functorial for maps of pairs of topological spaces.

There is also the dual form for cohomology of the Mayer-Vietoris sequence. In order to prove this result one needs to know that the inclusion $S_*^{\text{small}}(X) \to S_*(X)$ dualizes to a map $S^*(X) \to S^*_{\text{small}}(X)$ which induces an isomorphism on cohomology. This is an immediate consequence of theorem 4.1.1.

Theorem 4.1.6. (Mayer-Vietoris for Singular Cohomology) *Suppose $X = U \cup V$ with U, V open. Then we have the following long exact sequence,*

$$\cdots \longrightarrow H^{n+1}(U \cap V) \longrightarrow H^n(X) \xrightarrow{i_U^* - i_V^*} H^N(U) \oplus H^N(V) \xrightarrow{j_U^* + j_V^*} H^n(U \cap V) \longrightarrow \cdots$$

where $j_U : U \to X$, $j_V : V \to X$, $i_U : U \cap V \to U$ and $i_V : U \cap V \to V$ are the inclusions.

Proof. We have the short exact sequence of chain complexes,

$$0 \longrightarrow S_*(U \cap V) \xrightarrow{(i_U)_* - (i_V)_*} S_*(U) \oplus S_*(V) \xrightarrow{(j_U)_* + (j_V)_*} S_*^{\text{small}}(X) \longrightarrow 0$$

Dualizing, we obtain,

$$0 \longrightarrow S^*_{\text{small}}(X) \xrightarrow{(j_U)_*+(j_V)_*} S^*(U) \oplus S^*(V) \xrightarrow{(i_U)_*+-(i_V)_*} S^*(U \cap V) \longrightarrow 0$$

This gives rise to a long exact sequence in cohomology. Theorem 4.1.1, combined with the fact that the inclusion $S^{\text{small}}_*(X) \hookrightarrow S_*(X)$ induces an isomorphism on homology, implies that the dual map induces an isomorphism in cohomology, so we have the desired long exact sequence. □

The cohomological versions of excision and the dimension axiom are immediate,

Theorem 4.1.7. (Excision for Singular Cohomology) *Let (X, A) be a pair of topological spaces and $K \subset \bar{K} \subset \text{Int } A \subset A$. Then $H^*(X \setminus K; A \setminus K) \cong H^*(X; A)$.*

Theorem 4.1.8. (The Dimension Axiom for Cohomology)

$$H^*(\{pt\}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

4.1.1 Cup and Cap Product

One nice feature of singular cohomolgy, as opposed to singular homology, is that the singular cohomlogy of a space has a product structure which makes it into an associative graded ring with unit. Let X be any space, $S_*(X)$ be the singular chain complex associated to X and $S^*(X)$ the singular cochain complex. We define a bilinear product, *cup product*, $\cup : S^k(X) \otimes S^l(X) \to S^{k+l}(X)$. Suppose $\alpha \in S^k(X)$, $\beta \in S^l(X)$ and $(\sigma : \Delta^{k+l} \to X) \in S_{k+l}(X)$. For an n-simplex $\sigma : [v_0, \cdots, v_n] = \Delta^n \to X$ define $\text{fr}_i(\sigma) = \sigma|([v_0, \ldots, v_i] = \Delta^i) \to X$ and $\text{bk}_l(\sigma) = \sigma|([v_{n-l}, \ldots, v_n] = \Delta^l) \to X$. We say $\text{fr}_i(\sigma)$ is σ restricted to the *front* i simplex and $\text{bk}_l(\sigma)$ is σ restricted to the *back* l simplex. Then the cup product is given by

$$< \alpha \cup \beta, \sigma >=< \alpha, \text{fr}_k(\sigma) >< \beta, \text{bk}_l(\sigma) > .$$

Lemma 4.1.9. *Suppose $\alpha \in S^k(X)$, $\beta \in S^l(X)$, then*

$$\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta\beta.$$

Exercise 4.1.10. *Prove this lemma.*

Let $Z^n \subset S^n(X)$ denote the n-cocycles, and $B^n \subset S^n(X)$ denote the coboundaries.

Corollary 4.1.11. *If $\alpha \in Z^k$ and $\beta \in Z^l$ then $\alpha \cup \beta \in Z^{k+l}$.*
If $\gamma \in B^k$ and $\beta \in Z^l$ then $\gamma \cup \beta \in B^{k+l}$.
If $\alpha \in Z^k$ and $\gamma \in B^l$ then $\alpha \cup \gamma \in B^{k+l}$.

Proof. First suppose $\alpha \in Z^k$ and $\beta \in Z^l$. Then

$$\delta(\alpha \cup \beta) = 0 \cup \beta + (-1)^k \alpha \cup 0 = 0.$$

So $\alpha \cup \beta \in Z^{k+l}$.

Now, suppose $\gamma \in B^k$ and $\beta \in Z^l$. Let $\gamma = \delta\alpha$ for some $\alpha \in S^{k-1}(X)$. Then,

$$\gamma \cup \beta = \delta\alpha \cup \beta = \delta(\alpha \cup \beta) \in B^{k+l}.$$

Since $(-1)^k \alpha \cup \delta\beta = (-1)^k \alpha \cup 0 = 0$.

Similarly, suppose $\alpha \in Z^k$ and $\gamma \in B^l$. Let $\gamma = \delta\beta$ for some $\beta \in S^{l-1}(X)$. Then,

$$(-1)^k \alpha \cup \gamma = (-1)^k \alpha \cup \delta\beta = \delta(\alpha \cup \beta) \in B^{k+l}.$$

Since $\delta\alpha \cup \beta = 0 \cup \beta = 0$. □

Thus, we have a well defined product on cohomology, $\cup : H^k(X) \otimes H^l(X) \to H^{k+l}(X)$.

We can also define the adjoint to cup product, *cap product*. This is a bilinear product $\cap : S^l(X) \times S_{k+l}(X) \to S_k(X)$. It is adjoint to cup product in the sense that,

$$< \alpha, \beta \cap \sigma > = < \alpha \cup \beta, \sigma >$$

for $\alpha \in S^k(X)$, $\beta \in S^l(X)$ and $\sigma : \Delta^{k+l} \to X$. On the chain level the formula for cap product is,

$$\beta \cap \sigma = < \beta, \mathrm{bk}_l(\sigma) > \mathrm{fr}_k(\sigma).$$

One can easily check that with this formula cup and cap product are adjoints as desired.

Lemma 4.1.12. *Suppose $\beta \in S^l(X)$ and $\sigma : \Delta^{k+l} \to X$, then*

$$\partial(\beta \cap \sigma) = \beta \cap \partial\sigma - (-1)^k \delta\beta \cap \sigma$$

Proof. Let $\alpha \in S^k(X)$. Then,

$$\begin{aligned}
< \alpha, \partial(\beta \cap \sigma) > &= < \delta\alpha, \beta \cap \sigma > \\
&= < \delta\alpha \cup \beta, \sigma > \\
&= < \delta(\alpha \cup \beta), \sigma > -(-1)^k < \alpha \cup \delta\beta, \sigma > \\
&= < \alpha \cup \beta, \partial\sigma > -(-1)^k < \alpha, \delta\beta \cap \sigma > \\
&= < \alpha, \beta \cap \partial\sigma > -(-1)^k < \alpha, \delta\beta \cap \sigma > \\
&= < \alpha, \beta \cap \partial\sigma - (-1)^k \delta\beta \cap \sigma >
\end{aligned}$$

□

As in the cup product case, this formula shows that we have a well defined cap product, $\cap : H^l(X) \otimes H_{k+l}(X) \to H_k(X)$ via the following corollary.

Corollary 4.1.13. *If $\alpha \in Z^l$ and $\beta \in Z_{k+l}$ then $\alpha \cup \beta \in Z_k$.*
If $\gamma \in B^l$ and $\beta \in Z_{k+l}$ then $\gamma \cup \beta \in B_k$.
If $\alpha \in Z^l$ and $\gamma \in B_{k+l}$ then $\alpha \cup \gamma \in B_k$.

Exercise 4.1.14. *Prove this corollary.*

The cup product makes $H^*(X)$ into an associative graded ring with unit. The unit is the canonical generator $1 \in S^0(X)$, which evaluates $< 1, p >= 1$ on a point p. Then we have,

$$< 1 \cup \alpha, \sigma >=< 1, \mathrm{fr}_0(\sigma) >< \alpha, \mathrm{bk}_k(\sigma) >=< 1, p >< \alpha, (\sigma) >=< \alpha, (\sigma) > .$$

Also, cup product is natural in the following sense. If $f : Y \to X$ is a continuous map, then f induces $f^* : H^*(X) \to H^*(Y)$, and this map preserves cup product:

$$f^*[\alpha] \cup f^*[\beta] = [f^*\alpha] \cup [f^*\beta] = [f^*\alpha \cup f^*\beta],$$

as the following computation shows,

$$\begin{aligned} < f^*\alpha, \mathrm{fr}_k(\sigma) >< f^*\beta, \mathrm{bk}_l(\sigma) > &=< \alpha, f_*\mathrm{fr}_k(\sigma) >< \beta, f_*\mathrm{bk}_l(\sigma) > \\ &=< \alpha, \mathrm{fr}_k(f_*\sigma) >< \beta, \mathrm{bk}_l(f_*\sigma) > \\ &=< \alpha \cup \beta, f_*\sigma > . \end{aligned}$$

4.2 Ordered Simplicial (Co)Homology

Let K be a simplicial complex with $|K|$ its geometric realization. We define a chain complex with chain groups $C_k^{ord}(K)$ the free abelian group generated by $\{l : \Delta^k \to K|\ l \text{ is affine linear i.e. } l : V(\Delta^k) \to V(K) \text{ and } \mathrm{Im}(l) \in S(K)\}$. Geometrically these are maps $|l| : \Delta^k \to |K|$ such that the image is a simplex $\sigma \subset |K|$, that is we have $l : V(\Delta^k) \to V(\sigma)$ and then $|l| \sum a_i v_i = \sum a_i l(v_i)$ where $\Delta^k = \sum a_i v_i$ in barycentric coordinates. We define the boundary map $\partial : C_k^{ord}(K) \to C_{k-1}^{ord}(K)$ by

$$\partial l = \sum_{i=0}^{k} (-1)^i l|_{f_i}$$

where $f_i : \Delta^{k-1} \to \Delta^k$ is given by sending the first i vertices of Δ^{k-1} in barycentric coordinates. We define the boundary map $\partial : C_k^{ord}(K) \to C_{k-1}^{ord}(K)$ by

$$\partial l = \sum_{i=0}^{k} (-1)^i l|_{f_i}$$

where $f_i : \Delta^{k-1} \to \Delta^k$ is given by sending the first i vertices of Δ^{k-1} to the first i vertices of Δ^k and sending the last $k-i$ vertices of Δ^{k-1} to the last $k-i$ vertices of Δ^k. The usual computation shows that $\partial^2 = 0$.

The ordered simplicial chain groups of a simplicial complex sit naturally inside the singular chain groups of the geometric realization of that simplicial complex, $C_*^{ord}(K) \hookrightarrow S_*(|K|)$, and this inclusion commutes with the boundary map.

Theorem 4.2.1. *The inclusion $C_*^{ord}(K) \hookrightarrow S_*(|K|)$ induces an isomorphism on homology.*

Proof. ???? □ Give proof.

We have the dual ordered simplicial cochain complex as well as the versions of these chain complex and cochain complex with coefficients in any abelian group. These lead to the ordered simplicial homology and cohomology with coefficients in an abelian group.

There are also relative ordered chain groups and relative homology groups. Suppose that K is a simplicial complex and that L is a subcomplex. Then the inclusion map induces an inclusion

$$C_*^{\mathrm{ord}}(L) \to C_*^{\mathrm{ord}}(K).$$

Then we define $C_*^{\mathrm{ord}}(K, L) = C_*^{\mathrm{or}}(K)/C^{\mathrm{or}}(L)$. Its homology is the relative ordered homology of the pair (K, L). By the construction there is a long exact sequence of the pair for ordered simplicial homology.

Exercise 4.2.2. *Write down the long exact sequence of the pair and prove that it is exact. Show that this long exact sequence is natural for simplicial maps of pairs of simplicial complexes.*

4.3 Oriented Simplicial Homology and Cohomology

Let K be a simplicial complex (see appendix D). A simplex σ of degree i has two orientations. We can think of these orientations in two equivalent ways. First, any ordering of the vertices of σ determines an orientation of σ. Two orderings give the same orientation if the orderings differ by an even permuatation, and give the opposite orientation if they differ by an odd permutation. The second way to think of orientation is to take a linear embedding of σ into $\mathbb{R}^i$ given by mapping σ onto the standard i-simplex in $\mathbb{R}^{i+1}$, and then projecting to $\mathbb{R}^i$. Then an orientation of σ is given by the standard orientation of $\mathbb{R}^i$. Check that these two definitions of orientation are equivalent.

Now, we define the oriented simplicial chain groups, $C_i^{or}(K)$, to be a direct sum of infinite cyclic groups, one for each −-simplex of K. However, these summands are not canonically isomorphic to $\mathbb{Z}$, rather the summands are of the form $\mathbb{Z}[\sigma, o_\sigma] \oplus \mathbb{Z}[\sigma, -o_\sigma]/\{[\sigma, o_\sigma] = -[\sigma, o_\sigma]\}$ where o_σ is an orientation of σ. To choose a generator for the summand associated to σ is to choose an orientation of σ.

Definition 4.3.1. $C_i^{or} = \displaystyle\bigoplus_{\substack{\sigma \in K \\ |\sigma| = i}} \frac{\mathbb{Z}[\sigma, o_\sigma] \oplus \mathbb{Z}[\sigma, -o_\sigma]}{[\sigma, o_\sigma] \backsim -[\sigma, o_\sigma]}$

The boundary map $\partial : C_i^{or}(K) \to C_{i-1}^{or}(K)$ is given by

$$\partial([\sigma, o_\sigma]) = \sum_{\substack{\tau < \sigma \\ \#\tau = \#\sigma - 1}} [\tau, o_\sigma|_\tau].$$

An equivalent formula is given by

$$\partial[\sigma, \{v_0, \cdots, v_i\}] = \sum_{j=0}^{i} (-1)^j [\tau_j, \{v_0, \cdots, \hat{v}_j, \cdots, v_i\}].$$

where τ_j is the j-th codimension one face of σ. It should be clear from this formula that $\partial^2 = 0$. Check that the two formulas are in fact the same. So, $\{C_*^{or}(K), \partial\}$ is a chain complex, and we define the oriented simplicial homology of a simplicial complex to be the homology of this chain complex, $H_*^{or}(K) = H_*(\{C_*^{or}(K), \partial\})$.

The oriented simplicial cohomology is the cohomology of the dual cochain complex. It is denoted $H^*_{\mathrm{or}}(K)$. We also have the oriented simplicial homology and cohomology with coefficients in an abelian group A obtained by forming a chain complex by tensoring the given complex with A or obtained by forming a cochain complex by taking Hom of the complex into A. These are denoted $H_*^{\mathrm{or}}(K; A)$ and $H^*_{\mathrm{or}}(K; A)$, respectively.

If L is a subcomplex of K, then there is a natural inclusion $C_*^{\mathrm{or}}(L \to C_*^{\mathrm{or}}(K)$. We define the relative oriented chains $C_*^{\mathrm{or}}(K, L)$ to be the quotient complex, and we define the homology of this quotient complex to be the relative oriented simplicial homology, denoted $H_*^{\mathrm{or}}(K, L)$. By the construction, there is a long exact sequence of the pair for oriented simplicial homology.

Exercise 4.3.2. *Write down the long exact sequence of the pair and prove that it is exact. Show that this long exact sequence is natural for simplicial maps of pairs of simplicial complexes.*

4.4 Comparison of Oriented and Ordered Simplicial Homology

There is a natural chain map $C_*^{\mathrm{ord}}(K) \to C_*^{\mathrm{or}}(K)$. It is defined as follows. If $\sigma \colon \Delta^n \to K$ is an ordered n-simplex whose image is a simplex of dimension less than n, then σ is sent to zero in $C_*^{\mathrm{or}}(K)$. If on the other hand, the image of σ is of dimension n, then σ is an isomorphism from the standard n-simplex Δ^n to an n-simplex of K. Hence, it induces an orientation on this n-simplex. We associate to σ this induced oriented n-simplex.

Here is the theorem that compares ordered and oriented homology.

Theorem 4.4.1. *The map $C_*^{\mathrm{ord}}(K) \to C_*^{\mathrm{or}}(K)$ is a chain map. It induces an isomorphism on homology.*

Proof. We begin by showing that the map is a chain map. If $\sigma \colon \Delta^n \to K$ is an ordered n-simplex whose image has dimension less than $n-1$, then it is clear that σ and all its faces map to the zero element in $C^{\mathrm{or}}_*(K)$. Hence, on these elements the maps commute with the boundary maps. Suppose that the image of σ has dimension $n-1$. Then σ maps to zero in $C^{\mathrm{or}}_*(K)$. These means that exactly two of the vertices v_i, v_j of Δ^n are identified and the others are mapped to distinct vertices. All the faces of σ except the ones obtained by deleting v_i and v_j have images of dimension $n-2$ and hence map to zero in $C^{\mathrm{or}}_*(K)$. The remaining two faces map to oppositely oriented $n-1$ simplices and hence cancel in $C^{\mathrm{or}}_*(K)$. This shows that $\partial\sigma$ maps to zero in this case as well.

Lastly, it is clear when $\sigma\colon \Delta^n \to K$ is an embedding that the image of $\partial\sigma$ is equal to the boundary of the image of σ.

First we prove that for finite simplicial complexes, the induced map on homology is an isomorphism. This is done by induction on the number of simplices.

Claim 4.4.2. *Suppose that K is a single simplex and all its faces. Then the map $C^{\mathrm{ord}}_*(K, \partial K)$* ~~All~~ *$C^{\mathrm{or}}_*(K, \partial K)$ induces an isomorphism on homology.* its faces what?

Proof. Clearly, the oriented relative homology is a $\mathbb{Z}$ in degree equal to the dimension of the simplex and zero in all other degrees. The exercises below show that the ordered homology is isomorphic. □

Exercise 4.4.3. *Show that any relative ordered k-cycle of $(\Delta^n, \partial\Delta^n)$ is trivial if $k \neq n$ and any relative n-cycle is homologous to the identity map. Complete the proof of the previous claim.*

Now the argument for finite simplicial complexes goes by induction on the number of simplices using the above claim and the long exact sequences of the pairs for both of these homologies. For infinite complexes one uses the fact that homology comutes with direct limits (see Appendix B). □

Exercise 4.4.4. *Complete the proof of the theorem.*

Corollary 4.4.5. *The ordered simplicial homology of a simplicial complex is identified with the singular homology*

Proof. By Theorem **??** the oriented simplicial homology is identified with the singular homology and by Theorem 4.4.1 the ordered singular homology and the oriented singular homology are identified. □

4.5 DeRham Cohomology

In this section we will define a second cohomology theory, the DeRham cohomology of a smooth manifold. Eventually we will prove what is known as DeRham's theorem, which says that this cohomology agrees with the singular cohomology defined above for smooth manifolds.

Differential forms give a contravariant functor from the category of smooth manifolds and smooth maps to the category of real differential graded algebras.

$$M \mapsto \Omega^*(M) = \{\overset{dimM}{\underset{k=0}{\oplus}} \Omega^k(M), d\}$$

and,

$$(f : N \to M) \mapsto (f^* : \Omega^*(M) \to \Omega^*(N))$$

In particular, these differential graded algebras are cochain complexes, and so we can apply the cohomology functor. The composition of these functors gives a functor, the *DeRham cohomology functor*, from the category of smooth manifolds and smooth maps to the category of graded real vector spaces.

$$M \mapsto H^*_{dR}(M) = \text{cohomology of } \Omega^*(M).$$

In fact, we have more. The differential graded algebra structure descends to a graded algebra structure on cohomology. Given $[\alpha], [\beta] \in H^*_{dR}(M)$, choose closed form representatives α, β for these cohomology classes, and consider $\alpha \wedge \beta$. The Leibnitz rule tells us that this is a closed form. We define $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$. We need to check that this definition is independent of the closed form representatives that we use. So, if we have two other representatives, $\alpha + d\gamma \in [\alpha]$ and $\beta + d\mu \in [\beta]$, we need to show that,

$$[(\alpha + d\gamma) \wedge (\beta + d\mu)] = [\alpha \wedge \beta + \alpha \wedge d\mu + d\gamma \wedge \beta + d\gamma \wedge d\mu] = [\alpha \wedge \beta]$$

This follows immediately from the following lemma.

Lemma 4.5.1. *The wedge product of a closed from with an exact form is exact (and therefore an exact form wedge a closed form is exact).*

Proof. Suppose that α is a closed form i.e. $d\alpha = 0$, and $d\mu$ is an exact form. Then $d(-1)^{|\alpha|}(\alpha \wedge \mu) = \alpha \wedge d\mu$. □

So, $H^*_{dR}(M)$ is a graded comutative $\mathbb{R}$-algebra, with,

$$[\alpha] \wedge [\beta] = (-1)^{|\alpha||\beta|}[\beta] \wedge [\alpha]$$

.

4.5.1 Some Computations

Proposition 4.5.2.

$$H^*_{dR}(\{pt\}) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We have $\Omega^0 = f : \{pt\} \to \mathbb{R} \cong \mathbb{R}$ and $\Omega^k = 0$ for $k \neq 0$, and thus,

$$H^*_{dR}(\{pt\}) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

□

A less trivial computation is the DeRham cohomology of the circle.

Proposition 4.5.3.

$$H^*_{dR}(S^1) = \begin{cases} \mathbb{R} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We think of the circle as the quotient of the real line by translation by 2π.

$$S^1 = \mathbb{R}/(x \backsim x + 2\pi)$$

So the 0-forms are 2π periodic smooth functions on $\mathbb{R}$, and $\Omega^1(S^1) = \{f(t)dt \mid f : \mathbb{R} \to \mathbb{R} \text{ is smooth and } 2\pi \text{ periodic}\}$. There are no differential forms on S^1 of degree greater than or equal to two.

Now, $d(f(t)) = f'(t)dt$, so $\mathrm{Ker}(d) = \{\text{constant functions}\}$. Thus $H^0_{dR}(S^1) = \mathbb{R}$. Now, given $g(t)dt \in \Omega^1(S^1)$, the fundamental theorem of calculus tells us that there exists a function $f : \mathbb{R} \to \mathbb{R}$, unique up to a constannt, such that $f'(t) = g(t)$; however, f may not be periodic. The function f is obtained from g by integrating,

$$f(s) = \int_0^s g(t)dt.$$

So we see that f is periodic if and only if $\displaystyle\int_0^{2\pi} g(t)dt = 0$. We have a homomorphism $\int : \Omega^1(S^1) \to \mathbb{R}$ given by $\omega \mapsto \int_{S^1} \omega$, where we define integration over the S^1 by $\int_{S^1} g(t)dt = \displaystyle\int_0^{2\pi} g(t)dt$ where the latter is usual Riemannian integration. This is clearly an $\mathbb{R}$-homomorphism, and $\mathrm{Im}(d) = \mathrm{Ker}(\int)$. Thus, the map $\int$ is an isomorphism,

$$\Omega^1(S^1)/\mathrm{Im}(d) \xrightarrow[\cong]{\int} \mathbb{R}$$

and so we have,

$$H^*_{dR}(S^1) = \begin{cases} \mathbb{R} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

□

Exercise 4.5.4. *Compute the deRham cohomology of the two torus, $\mathbb{R}^2/\mathbb{Z}$.*

4.6 Čech Cohomology

Čech cohomology, like singular cohomology, can be defined for any topological space X. Initially, we define the Čech cohomology with respect to an open cover of X.

Definition 4.6.1. Let X be a topological space and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. The degree k *Čech cochain group* of X with respect to this open cover is defined to be

$\check{C}^k(X; \{U_\alpha\}_{\alpha\in A}) = \{\phi : \{\text{ordered } k{+}1 \text{ tuples of } A\} \to \mathbb{Z} | \phi \text{ satisfies properties 1 and 2 below}\}$

1. $\phi(U_{\alpha(\sigma(0))}, \ldots, U_{\alpha(\sigma(k))}) = (-1)^{|\sigma|}\phi(U_{\alpha(0)}, \ldots, U_{\alpha(k)})$ for $\sigma \in \Sigma_{k+1}$.
2. $\phi(U_{\alpha(0)}, \ldots, U_{\alpha(k)}) = 0$ if $U_{\alpha(0)} \cap \ldots \cap U_{\alpha(k)} = \emptyset$.

If we fix a well ordering of A and consider the free abelian group C_k generated by $\{\alpha(0) < \cdots < \alpha(k) | U_{\alpha(0)} \cap \ldots \cap U_{\alpha(k)} \neq \emptyset\}$ then $\check{C}^k = \mathrm{Hom}(C_k; \mathbb{Z})$. Thus, we see that like the singular cochain groups, the Čech cochain groups are dual to free abelian groups.

Now we define the coboundary map $\delta : \check{C}^k(X; \{U_\alpha\}) \to \check{C}^{k+1}(X; \{U_\alpha\}$ by,

$$\delta(\phi)(U_{\alpha(0)}, \ldots, U_{\alpha(k+1)}) = \sum_{i=0}^{k+1}(-1)^i\phi(U_{\alpha(0)}, \ldots, \widehat{U}_{\alpha(i)}, \ldots, U_{\alpha(k+1)})$$

This definition is understood to hold only in the case where $U_{\alpha(0)} \cap \ldots \cap U_{\alpha(k)} \neq \emptyset$. If the intersection is empty then $\delta(\phi)(U_{\alpha(0), \ldots, U_\alpha(k+1)}) = 0$. Since δ is linear, it is an abelian group homomorphism. Next we check that $\delta^2 = 0$, so we have defined a cochain complex. Symbolically, the computation is very similar to the computations made for our previous cochain complex constructions.

Proposition 4.6.2. $\delta^2 = 0$

Proof. Assume that $U_{\alpha(0)} \cap \ldots \cap U_{\alpha(k+2)} \neq \emptyset$.

Then,

$$\begin{aligned}
\delta(\delta\phi)(U_{\alpha(0)},\dots,U_{\alpha(k+2)}) &= \sum_{i=0}^{k+2}(-1)^i\delta\phi(U_{\alpha(0)},\dots,\widehat{U}_{\alpha(i)},\dots,U_{\alpha(k+2)}) \\
&= \sum_{i=0}^{k+2}(-1)^i\Bigg[\sum_{j=0}^{i-1}(-1)^j\phi(U_{\alpha(0)},\dots,\widehat{U}_{\alpha(j)},\dots,\widehat{U}_{\alpha(i)},\dots,U_{\alpha(k+2)}) \\
&\quad + \sum_{j=i+1}^{k+2}(-1)^{j-1}\phi(U_{\alpha(0)},\dots,\widehat{U}_{\alpha(i)},\dots,\widehat{U}_{\alpha(j)},\dots,U_{\alpha(k+2)})\Bigg] = 0
\end{aligned}$$

Since in this sum, for a given pair $a < b$ each term $\phi(U_{\alpha(0)},\dots,\widehat{U}_{\alpha(a)},\dots,\widehat{U}_{\alpha(b)},\dots,U_{\alpha(k+2)})$ appears exactly twice and with cancelling signs. □

Thus $(\check{C}^*(X;\{U_\alpha\}),\delta)$ forms a cochain complex. We define the *Čech cohomology* of X with respect to the open cover $\{U_\alpha\}$ to be the cohomology of this cochain complex, $\check{H}^*(X;\{U_\alpha\}) = H^*(\check{C}^*(X;\{U_\alpha\}),\delta)$.

4.6.1 Some Computations

Proposition 4.6.3. *Let X be any topological space and let $\{X\}$ be the open cover consisting of the single open set X. Then,*

$$\check{H}^*(X;\{X\}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since there is only one set in our open cover, $\check{C}^*(X;\{X\}) = 0$ for $* \neq 0$ and $\check{C}^0(X;\{X\}) = \mathbb{Z}$. □

Proposition 4.6.4. *Let X be any topological space and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover consisting of connected open sets. Then,*

$$\check{H}^0(X;\{U_\alpha\}_{\alpha\in A}) = \{\phi : \{\textit{components of } X\} \to \mathbb{Z}\}$$

Proof. We have $\check{C}^0(X;\{U_\alpha\}_{\alpha\in A}) = \mathbb{Z}^A$. Suppose that $\phi : A \to \mathbb{Z}$ is in $\check{C}^0(X;\{U_\alpha\}_{\alpha\in A})$. Then

$$\delta\phi(U_\alpha,U_\beta) = \begin{cases} 0 & U_\alpha \cap U_\beta = \emptyset \\ \phi(\alpha)-\phi(\beta) & U_\alpha \cap U_\beta \end{cases}$$

Thus,

$$\check{H}^0(X;\{U_\alpha\}_{\alpha\in A}) = \{\phi : A \to \mathbb{Z} | \phi(\alpha) = \phi(\beta) \text{ if } U_\alpha \cap U_\beta \neq \emptyset\} = \{\phi : \{\text{components of } X\} \to \mathbb{Z}\}$$

□

Proposition 4.6.5. *Let $\{I_j\}_{j=1}^t$ be an open cover of S^1 by t open intervals so that each interval $I_i \cap I_j = \emptyset$ unless $i = j, j+1, j-1$ in cyclic order i.e. $I_{t+1} = I_1$. Then,*

$$\check{H}^*(S^1; \{I_j\}_{j=1}^t) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 4.6.6. *Prove this proposition.*

4.6.2 Dependance on the Open Cover

Now we would like to remove the dependance of the Čech cohomology on the choice of an open cover, so we have a topological invariant. Suppose that we have two open covers of X, $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$. We say that $\{V_\beta\}_{\beta\in B}$ *refines* $\{U_\alpha\}_{\alpha\in A}$ if there is a function $r : B \to A$ so that $V_\beta \subset U_{r(\beta)}$ for every $\beta \in B$. Such a function is called a *refinement function*.

Theorem 4.6.7. *Suppose that $\{V_\beta\}_{\beta\in B}$ refines $\{U_\alpha\}_{\alpha\in A}$. Let $r : B \to A$ be a refinement function. Then r determines a cochain map $r^* : \check{C}^*(X; \{U_\alpha\}) \to \check{C}^*(X; \{V_\beta\})$. The induced map on cohomology is independent of the choice of refinement function r.*

Proof. Let $\phi \in \check{C}^k(X; \{U_\alpha\})$. Then we define $r^*\phi$ by,

$$r^*\phi(V_{\beta(0)}, \ldots, V_{\beta(k)}) = \phi(U_{r\beta(0)}, \ldots, U_{r\beta(k)})$$

assuming that $V_{\beta(0)} \cap \ldots \cap V_{\beta(k)} \neq \emptyset$, in which case of course $r^*\phi(V_{\beta(0)}, \ldots, V_{\beta(k)}) = 0$. Then $r^*\phi \in \check{C}^*(X; \{V_\beta\})$. We claim that this defines a cochain map. We have,

$$\begin{aligned}
\delta(\delta(r^*\phi))((V_{\beta(0)}, \ldots, V_{\beta(k)}) &= \sum_{i=0}^{k+1} (-1)^i r^*\phi(V_{\beta(0)}, \ldots, \widehat{V}_{\beta(i)}, \ldots, V_{\beta(k+1)}) \\
&= \sum_{i=0}^{k+1} (-1)^i \phi(U_{r\beta(0)}, \ldots, \widehat{U}_{r\beta(i)}, \ldots, U_{r\beta(k+1)}) \\
&= \delta\phi(U_{r\beta(0)}, \ldots, U_{r\beta(k)}) \\
&= r^*\delta\phi((V_{\beta(0)}, \ldots, V_{\beta(k)})
\end{aligned}$$

Thus, we have an induced map on cohomology, $r^*\check{H}^*(X; \{U_\alpha\}) \to \check{H}^*(X; \{V_\beta\})$. Now, suppose that $s : B \to A$ is another refinement map. We use r and s to define a cochain homotopy $H_{(r,s)} : \check{C}^k(X, \{U_\alpha\}) \to \check{C}^{k-1}(X, \{V_\beta\})$ such that $\delta H_{(r,s)} + H_{(r,s)}\delta = s^* - r^*$, and thus $r^* = s^*$ on cohomology.

Given $\phi \in \check{C}^k(X, \{U_\alpha\})$ define

$$H_{(r,s)}\phi(V_{\beta(0)}, \ldots, V_{\beta(k-1)}) = \sum_{i=0}^{k-1} (-1)^i \phi(U_{r\beta(0)}, \ldots, U_{r\beta(i)}, U_{s\beta(i)}, \ldots, U_{s\beta(k-1)})$$

Then we have,

$$
\begin{aligned}
(\delta H)(\phi)(V_{\beta(0)},\dots,V_{\beta(k)}) &= \sum_{i=0}^{k}(-1)^i H\phi(V_{\beta(0)},\dots,\widehat{V}_{\beta(i)},\dots,V_{\beta(k)})\\
&= \sum_{i=0}^{k}(-1)^i\Bigg[\sum_{j=0}^{i-1}(-1)^j\phi(U_{r\beta(0)},\dots,U_{r\beta(j)},U_{s\beta(j)},\dots,\widehat{U}_{s\beta(i)},\dots,U_{s\beta(k)})\\
&\quad+\sum_{j=i+1}^{k}(-1)^{j-1}\phi(U_{r\beta(0)},\dots,\widehat{U}_{r\beta(i)},\dots,U_{r\beta(j)},U_{s\beta(j)},\dots,U_{s\beta(k)})\Bigg].
\end{aligned}
$$

and,

$$
\begin{aligned}
(H\delta)(\phi)(V_{\beta(0)},\dots,V_{\beta(k)}) &= \sum_{j=0}^{k}(-1)^j\delta\phi(U_{r\beta(0)},\dots,U_{r\beta(j)},U_{s\beta(j)},\dots,U_{s\beta(k)})\\
&= \sum_{j=0}^{k}(-1)^j\Bigg[\sum_{i=0}^{j}(-1)^i\phi(U_{r\beta(0)},\dots,\widehat{U}_{r\beta(i)},\dots,U_{r\beta(j)},U_{s\beta(j)},\dots,U_{s\beta(k)})\\
&\quad+\sum_{i=j}^{k}(-1)^{i+1}\phi(U_{r\beta(0)},\dots,U_{r\beta(j)},U_{s\beta(j)},\dots,\widehat{U}_{s\beta(i)},\dots,U_{s\beta(k)})\Bigg].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(\delta H+H\delta)\phi(V_{\beta(0)},\dots,V_{\beta(k)}) = \sum_{j=0}^{k}(-1)^j\Bigg[&(-1)^j\phi(U_{r\beta(0)},\dots,U_{r\beta(j-1)},U_{s\beta(j)},\dots,U_{s\beta(k)})\\
&+(-1)^{j+1}\phi(U_{r\beta(0)},\dots,U_{r\beta(j)},U_{s\beta(j+1)},\dots,U_{s\beta(k)})\Bigg]
\end{aligned}
$$

This sum telescopes leaving,

$$
(\delta H+H\delta)\phi(V_{\beta(0)},\dots,V_{\beta(k)}) = \phi(U_{s\beta(0)},\dots,U_{s\beta(k)}) - \phi(U_{r\beta(0)},\dots,U_{r\beta(k)})
$$

□

Now, let X be a topological space, and let $\mathcal{O}$ be the set of all open covers of X. We make $\mathcal{O}$ into a directed set using the partial order $\{U_\alpha\}_{\alpha\in A}\le\{V_\beta\}_{\beta\in B}$ if $\{V_\beta\}_{\beta\in B}$ refines $\{U_\alpha\}_{\alpha\in A}$. You can easily check that this relation makes $\mathcal{O}$ a poset. To see that it is directed, given two open covers $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$, the open cover $\{U_\alpha\cap V_\beta\}_{(\alpha,\beta)\in A\times B}$ refines both of them.

For any open cover $o\in\mathcal{O}$ we have $\check{H}^*(X;o)$, and if $o\le o'$ we have a map $r^*_{o,o'}$: $\check{H}^*(X;o)\to\check{H}^*(X;o')$. Furthermore, if $o\le o'\le o''$ are open covers of X, then $r^*_{o',o''}\circ r^*_{o,o'}=r^*_{o,o''}$. Thus, $\{\check{H}^*(X,o),r^*_{o,o'}\}_{o\in\mathcal{O}}$ is a directed system of graded abelian groups and graded group homomorphisms.

Definition 4.6.8. The *Čech cohomology groups* of a space X are defined to be $\check{H}^*(X) = \varinjlim_{o \in \mathcal{O}} \{\check{H}^*(X, o), r^*_{o,o'}\}$

Theorem 4.6.9. *$\check{H}^*(S^1) \cong \check{H}^*(S^1; \{I_j\}_{j=1}^t)$ where $\{I_j\}_{j=1}^t$ is the open cover of proposition 4.6.5 and the isomorphism is induced by the natural inclusion.*

Proof. Covers of this type are cofinal in the set of all open covers of S^1. Now use compactness and a Lebesgue number argument along with the following claim to complete the proof.

Claim 4.6.10. *If $\{I_j\}_{j=1}^n \le \{I'_l\}_{l=1}^m$ then $\check{H}^*(S^1; \{I_j\}_{j=1}^n) \cong \check{H}^*(S^1; \{I'_l\}_{l=1}^m)$, where the isomorphim is the map induced by any refinement map.*

□

4.6.3 Connection with oriented simplicial cohomology

We will show that the Čech cohomology of a space with respect to an open cover is the oriented simplicial cohomology of some simplicial complex associated to the open cover. Given a topological space X and an open cover $\{U_\alpha\}_{\alpha \in V}$, we define a simplicial complex $K = \mathrm{Nerve}(\{\mathrm{U}_\alpha\}_{\alpha \in \mathrm{V}})$ called the *nerve of the open cover*. We assume that the open cover does not contain the empty set. The vertices of K are given by the index set of the open cover, V. Then $S \in 2^V \setminus \{\emptyset\}$ is a simplex of K iff $|S| < \infty$ and $\bigcap_{\alpha \in S} U_\alpha \neq \emptyset$.

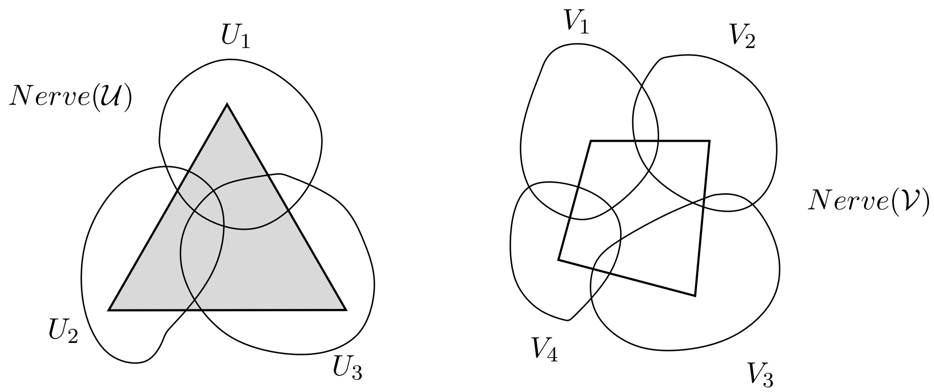

Figure 13: The nerves of two open covers

Now, suppose that we have two open covers $\{U_\alpha\}_{\alpha \in V}$ and $\{V_\beta\}_{\beta \in V'}$ of X, such that $\{V_\beta\}_{\beta \in V'}$ is a refinement of $\{U_\alpha\}_{\alpha \in V}$. Let $r : V' \to V$ be a refinement map. Then for a simplex $S' \subset 2^{V'}$ in $\mathrm{Nerve}(\{V_\beta\}_{\beta \in V'})$, we see that $r(s') \subset 2^V$ is a simplex in $\mathrm{Nerve}(\{U_\alpha\}_{\alpha \in V})$. Thus, r induces a map between the nerves of the open covers and therefore induces a map $r^* : H^*_{or}(\mathrm{Nerve}(V)) \to H^*_{or}(\mathrm{Nerve}(V'))$.

Theorem 4.6.11. *The Čech cochains with respect to an open cover $\mathcal{U}$ are identified with $(C^*_{\rm or}(\text{Nerve}(\mathcal{U}))$.*

Proof. This is true at the chain level, and the boundary maps are exactly the same. Consider $\rho \in C^i_{or}(\text{Nerve}(\mathcal{U})$. Then ρ evaluates on $[\sigma, o_\sigma]$ to give an integer, where σ is an i-simplex of Nerve($\mathcal{U}$), with the property $\rho([\sigma, o_\sigma]) = -\rho([\sigma, -o_\sigma]$. An element $\phi \in \check{C}^i(X;\mathcal{U})$ evaluates on an i-tuple of open sets in $\mathcal{U}$ to give an integer with the properties that $\phi(U_{\alpha(1)}, \ldots, U_{\alpha(i)}) = 0$ if $\cap_{j=1}^i U_{\alpha(j)} = \emptyset$, and if we change the order of the open sets by a permutation then the value of ϕ changes by the sign of that permutation, i.e. $\phi(U_{\alpha(1)}, \ldots, U_{\alpha(i)}) = \text{sign}(\pi)\phi(U_{\alpha(\pi(1))}, \ldots, U_{\alpha(\pi(i))})$ for $\pi \in \Sigma_{n+1}$. $\square$

Corollary 4.6.12. *The Čech cohomology of an open cover is the singular cohomology of the geometric realization of the nerve of the open cover.*

Theorem 4.6.13. *Let K be a simplicial complex. The Čech cohomology of K is identified with the singular cohomology of $|K|$ in a manner compatible with simplicial mappings.*

Proof. Given a simplicial complex K we define an open covering $\{U_v\}$ of $|K|$ whose open sets are indexed by the vertices of K. For a vertex v of K we consider the open star U_v. This is the union of all open simplices of $|K|$ whose closures contain v. It is easy to see that U_v is an open subset of $|K|$ and that the U_v give an open covering of $|K|$. Furthermore, $U_{v_1} \cap \cdots \cap U_{v_k}$ is non-empty if and only if the vertices $v_1, \ldots, v_k$ span a simplex σ of K. Thus, the nerve of this covering is identified with the original simplicial complex K. Hence, the Čech cohomology of this open covering is identified with the simplicial cohomology of K.

Now the open coverings of $|K|$ that arise from the above construction applied to simplicial subdivisions of K form a cofinal set of open coverings of $|K|$. The cohomology of each of these coverings is identified with the cohomology of the simplicial complex and hence with the singular cohomology of $|K|$. These identifications are compatible with refinement maps of the open coverings, proving that the Čech cohomology of $|K|$ is identified with the singular cohomology of $|K|$.

If $\varphi\colon L \to K$ is a simplicial mapping, then it induces a continuous mapping $|\varphi|\colon |L| \to |K|$. The open covering of $|K|$ by open stars of vertices of K pulls back to an open covering of $|L|$ which is refined by the open covering of $|L|$ by the open stars of its vertices. Hence, there is an induced mapping between the nerves of these coverings which is clearly φ. Thus, the map induced on Čech cohomology by $|\varphi|$ agrees with the map induced by φ on simplicial cohomology, and hence the map induced by $|\varphi|$ on singular cohomology. $\square$

Exercise 4.6.14. *Establish that the open stars U_v are open subsets of $|K|$ and their union as v ranges over all vertices of K is an open covering of $|K|$. Show that $U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset$ if and only if $v_1, \ldots, v_k$ span a simplex of K. Show that the collection of all such open* *coverings of $|K|$ associated to all subdivisions of K form a cofinal sequence of open covers of $|K|$. Lastly, show that if K' is a subdivision of K and if $\{U_{v'}\}$ and $|\{U_v\}$ are the open coverings associated to these two simplicial complexes, then a refinement map is determined by a simplicial map $K' \to K$ which sends each vertex v' of K' to some vertex of the closed simplex of $|K|$ that contains v'. Show that any such simplicial map induces a continuous map on the geometric realizations $|K'| \to |K|$ that is homotopic to the identity. Show also that the induced mapping on Čech cohomology is compatible with the given identifications of the Čech cohomology with the simplicial cohomology.*

4.6.4 The Axioms for Čech Cohomology

Dimension Axiom: We have already seen that the Čech cohomology of a point is a $\mathbb{Z}$ in degree zero and zero in all other degrees, so the dimension axiom holds for Čech cohomology.
Relative Čech Cohomology and the Long Exact Sequence of a Pair: If (X, A) is a pair of spaces, then any open covering of X induces one of A with index set consisting of the open subsets in the covering of X that meet A. There is clearly an inclusion of the nerve of the induced open covering of A as a subcomplex of the nerve of the open covering of X, and hence an induced surjective mapping from the Čech cochains of X with respect to this open covering to the Čech cochains of A with respect to the induced open covering of A. The relative Čech cochains of (X, A) with respect to the given open covering are the complex of the kernels of these surjective restriction mappings. With this definition, we have a long exact sequence of the pair for the Čech cohomology with respect to the given open covering.

As we pass from a covering to a refinement, this construction is natural. Taking the direct limit defines the relative Čech cohomology of the pair. Since homology commutes with direct limits, it is easy to see that there is a long exact sequence of the pair in Čech cohomology that is natural with respect to maps of pairs.
Mayer-Vietoris:
Excision:
Homotopy Axiom:

4.7 Group Cohomology

Let G be a group, A an abelian group and $G \times A \to A$ an action i.e. $\rho : G \to \mathrm{Aut}(A)$. We define a cochain complex with the k-th cochain group the set of all functions from k-tuples of elements in G to A, $C^k(G; A) = \{\phi : G \times \cdots \times G \to A\}$. These make a group under addition. The coboundary map is given by

$$\delta(\phi)(g_0, \ldots, g_k) = g_0\phi(g_1, \ldots, g_k) + \sum_{i=1}^{k+1}(-1)^i\phi(g_0, \ldots, (g_{i-1}g_i), g_{i+1}, \ldots, g_k),$$

where by convention the $k+1$ term in the sum is $(-1)^{k+1}\phi(g_0, \ldots, g_{k-1})$.

To give us some intuition for the definition of this coboundary map we first think of a group as a category, and then given a category we associate a quotient of a simplicial complex. We will show that the coboundary map given above corresponds closely to the usual boundary of this geometric object. Given a group G, the group can be thought of as a category with a single object $\{*\}$, and $\mathrm{Hom}(\{*\}, \{*\}) = G$. Given a category we associate a geometric object X_G with a vertex for every object in the category and an oriented edge for each morphism starting at the initital object (i.e. the domain) of the morphism and ending at the terminal object (i.e. the range) of the morphism. X_G contains a 2-simplex everytime we have three edges making a commutative triangle. For higher dimensional simplices, every time X_G contains the boundary of an n-simplex, then the n-simplex is also included.

In this geometric construction an n-simplex represents a chain of compositions,

$$\cdot \xrightarrow{g_1} \cdot \xrightarrow{g_2} \cdot \xrightarrow{g_3} \cdots \xrightarrow{g_n} \cdot$$

If we consider the usual boundary of this simplex, $\partial(\Delta^n) = f_0 - f_1 + f_2 - \cdots \pm f_n$, we see that this corresponds to the coboundary map given above, with the only change being the additional action of g_0 on the first term. This makes it clear that $\delta^2 = 0$, and we have defined a cochain complex. We define $H^*(G; A)$ to be the cohomology of this cochain complex.

Lets examine these cohomology groups in low dimensions. The 0-cocycles are functions from 0-tuples of elements in G to A, i.e. just elements in A, with the property that $\delta(a)(g) = ga - a = 0$ for $a \in A,\ g \in G$. So $\delta(a) = 0$ iff $a \in A^G$, the set of elements fixed under the action of G. Thus, $H^0(G; A) = A^G$.

Now lets consider the 1-cochains. These are maps $\phi : G \to A$ with $\delta(\phi)(g_0, g_1) = g_0\phi(g_1) - \phi(g_0g_1) + \phi(g_0) = 0$. That is $\phi(g_0g_1) = g_0\phi(g_1) + \phi(g_0)$. Such a map is called a *crossed homomorphism*. Thus, $H^1(G; A)$ is the crossed homomorphisms modulo the trivial crossed homomorphisms, $\phi_a(g) = ga - a$.

4.7.1 Group Cohomology and Group Extensions

Let A be an abelian group and $G \times A \to A$ an action of G with values in A. We wish to classify exact sequences

$$\{1\} \longrightarrow A \longrightarrow H \longrightarrow G \longrightarrow \{1\}$$

of G by A with the proviso that the action of G on A is the given one. That is, the action of G on A given by lifting an element $g \in G$ to $h_g \in H$ and then conjugating, $a \mapsto h_g a h_g^{-1}$ is the given action. The notion of isomorphism is the natural one – isomorphism of the middle extension groups that induces the identity of A to itself and induces the identity on the quotient G.

$$\begin{tikzcd}
\{1\} \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & H \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & \{1\} \\
\{1\} \arrow[r] & A \arrow[r] & H' \arrow[r] & G \arrow[r] & \{1\}
\end{tikzcd}$$

Given such an extension ext we define a 2-cochain on G with values in A by choosing arbitrarily for each $g \in G$ liftings $h_g \in H$ projecting to $g \in G$. We do this so that the lifting of the identity element is the identity element of H and so that $h_{g^{-1}} = (h_g)^{-1}$. Then

$$ext(g_1, g_2) = h_{g_1} h_{g_2} (h_{g_1 g_2})^{-1}.$$

We claim that ext is a cocycle. For this we compute:

$$\begin{aligned}
\delta(ext)(g_1, g_2, g_3) &= g_1 ext(g_2, g_3) - ext(g_1 g_2, g_3) + ext(g_1, g_2 g_3) - ext(g_1, g_2) \\
&= \left(h_{g_1} h_{g_2} h_{g_3} h_{g_2 g_3}^{-1} h_{g_1}^{-1}\right)\left(h_{g_1 g_2} h_{g_3} h_{g_1 g_2 g_3}^{-1}\right)^{-1}\left(h_{g_1} h_{g_2 g_3} h_{g_1 g_2 g_3}^{-1}\right)\left(h_{g_1} h_{g_2} h_{g_1 g_2}^{-1}\right)^{-1} \\
&= \left(h_{g_1} h_{g_2} h_{g_3} h_{g_2 g_3}^{-1} h_{g_1}^{-1}\right)\left(h_{g_1} h_{g_2 g_3} h_{g_1 g_2 g_3}^{-1}\right)\left(h_{g_1 g_2 g_3} h_{g_3}^{-1} h_{g_1 g_2}^{-1}\right)\left(h_{g_1 g_2} h_{g_2}^{-1} h_{g_1}^{-1}\right) \\
&= 1 \in H
\end{aligned}$$

We made a choice of our lift in H. Suppose we vary this choice by $h_g \rightsquigarrow \psi(g) h_g$ for some $\psi(g) \in A$. How does this affect our cocycle? We had $ext(g_1, g_2) = h_{g_1} h_{g_2} (h_{g_1 g_2})^{-1}$. Now, we have,

$$ext'(g_1, g_2) = \psi(g_1) h_{g_1} \psi(g_2) h_{g_2} (\psi(g_1 g_2) h_{g_1 g_2})^{-1}$$

Notice that $h_{g_1} \psi(g_2) = (h_{g_1} \psi(g_2) h_{g_1}^{-1}) h_{g_1} = g_1 \psi(g_2) h_{g_1}$. So,

$$\begin{aligned}
ext'(g_1, g_2) &= \psi(g_1) g_1 \psi(g_2) h_{g_1} h_{g_2} h_{g^{-1} g_2}^{-1} \psi(g_1 g_2)^{-1} \\
&= \psi(g_1) g_1 \psi(g_2) \psi(g_1 g_2)^{-1} ext(g_1, g_2)
\end{aligned}$$

which we can write additively as,

$$ext(g_1, g_2) + \delta ext(g_1, g_2).$$

So, a choice of lift determines a cocycle, and varying the lifts adds a coboundary. Thus the invariant of the extension is a class $[ext(H)] \in H^2(G; A)$. We call this the *extension clas*. Furthermore, if two extensions are isomorphic, then the extension classes are equal in $H^2(G; A)$.

Theorem 4.7.1. *$H^2(G; A)$ classifies extensions up to isomorphism, i.e. two extensions are isomorphic if and only if they have the same extension class and every cohomology class occurs as an extension class.*

Proof. Suppose we have two extensions, H and F such that $[ext(H)] = [ext'(F)] \in H^2(G; A)$. Choose lifts $\{h_g\}_{g \in G} \in H$ and $\{f_g\}_{g \in G} \in F$ that give $ext, ext' \in Z^2(G; A)$ respectively. Our hypothesis that $[ext] = [ext'] \in H^2(G; A)$ implies that $ext' = ext + \delta\psi$ for some $\psi : G \to A$. Now, we use $\{\psi(g) h_g\}_{g \in G}$ as lifts. With this new choice of lifts,

$ext = ext' \in Z^2(G;A)$. Define a map $\rho : H \to F$ by $ah_g \mapsto af_g$. We will show that this map is a homomorphism, and thus an isomorphism (by the five-lemma). First,

$$\begin{aligned}\rho[(ah_{g_1})(bh_{g_2})] &= \rho(a(g_1b)h_{g_1}h_{g_2})\\ &= \rho(a(g_1b)ext(g_1,g_2)h_{g_1g_2})\\ &= a(g_1b)ext(g_1,g_2)f_(g_1g_2)\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}\rho(ah_{g_1})\rho(bh_{g_2}) &= af_{g_1}bf_{g_2}\\ &= a(g_1b)f_{g_1}f_{g_2}\\ &= a(g_1b)ext(g_1,g_2)f_{g_1g_2}\end{aligned}$$

And, thus, $\rho[(ah_{g_1})(bh_{g_2})] = \rho(ah_{g_1})\rho(bh_{g_2})$. So, we have shown that two extensions are isomorphic if and only if they have the same extension class. Now we show that every cohomology class occurs as an extension class.

Given $\varphi \in Z^2(G;A)$ we want to give a group structure to the set $\Gamma_\varphi = \{(a,g)|a \in A, g \in G\}$. We define the multiplication map as follows, writing additively in A,

$$(a,g_1)(b,g_2) = (a + (g_1b) + \varphi(g_1,g_2), g_1g_2)$$

We claim that this gives a group structure on Γ_φ. The identity element is $(-\varphi(e,e),e)$. In order to show this, we first show that $\varphi(e,g)$ is independent of g.

$$\begin{aligned}\delta\varphi(e,g_1,g_2) &= 0\\ &= e(\varphi(g_1,g_2)) - \varphi(g_1,g_2) + \varphi(e,g_1g_2) - \varphi(e,g_1)\end{aligned}$$

And thus,

$$ext(e,g_1g_2) = \varphi(e,g_1).$$

Now to check that $(-\varphi(e,e),e)$ acts as the identity,

$$\begin{aligned}(-\varphi(e,e),e)(a,g) &= (-\varphi(e,e) + (ea) + \varphi(e,g), eg)\\ &= (a - \varphi(e,e) + \varphi(e,g), g)\\ &= (a,g)\end{aligned}$$

Now, we show that this multiplicative structure is associative.

$$\begin{aligned}\Big((a,g_1)(b,g_2)\Big)(c,g_3) &= (a + (g_1b) + \varphi(g_1,g_2), g_1g_2)(c,g_3)\\ &= (a + (g_1b) + \varphi(g_1,g_2) + (g_1g_2)c + \varphi(g_1g_2,g_3), (g_1g_2)g_3)\end{aligned}$$

and,

$$\begin{aligned}(a,g_1)\Big((b,g_2)(c,g_3)\Big) &= (a,g_1)(b+(g_2c)+\varphi(g_2,g_3),g_2g_3)\\ &= (a+(g_1((b+(g_2c)+\varphi(g_2,g_3))+\varphi(g_1,g_2g_3),g_1(g_2g_3))\\ &= (a+(g_1b)+g_1(g_2c)+g_1\varphi(g_2,g_3)+\varphi(g_1,g_2g_3),g_1(g_2g_3))\end{aligned}$$

By the associativity of G, $(g_1g_2)g_3 = g_1(g_2g_3)$, so we need to show,

$$a+(g_1b)+\varphi(g_1,g_2)+(g_1g_2)c+\varphi(g_1g_2,g_3) = a+(g_1b)+g_1(g_2c)+g_1\varphi(g_2,g_3)+\varphi(g_1,g_2g_3),g_1(g_2g_3)).$$

So, it is enough to show that,

$$\varphi(g_1,g_2)+\varphi(g_1g_2,g_3) = g_1\varphi(g_2,g_3)+\varphi(g_1,g_2g_3).$$

But this is exactly the cocycle condition for φ,

$$\delta(\varphi)(g_1,g_2,g_3) = g_1\varphi(g_2,g_3)-\varphi(g_1g_2,g_3)+\varphi(g_1,g_2g_3)-\varphi(g_1,g_2) = 0.$$

Finally, to find inverses, suppose that $(a,g)(b,g^{-1}) = (-\varphi(e,e),e)$. Then,

$$(a+(gb)+\varphi(g,g^{-1}),e) = (-\varphi(e,e),e).$$

So,

$$(a+(gb)+\varphi(g,g^{-1}) = -\varphi(e,e).$$

And thus,

$$b = g^{-1}\Big(-\varphi(e,e)-\varphi(g,g^{-1})-a\Big).$$

So, Γ_φ is a group and we have the extension,

$$\{1\} \longrightarrow A \longrightarrow \Gamma_\varphi \longrightarrow G \longrightarrow \{1\}$$

Finally, we claim that $[ext(\Gamma_\varphi)] = [\varphi] \in H^2(G;A)$. □

4.7.2 Group Cohomology and Representation Varieties

Let $\rho : G \to \mathrm{Aut}(V^n)$ be a representation of G. Then we have the adjoint representation $ad(\rho) : G \to \mathrm{Aut}(\mathrm{End}(V^n))$ given by

$$ad(\rho)(g)(\alpha) = (\rho(g))\alpha(\rho(g)^{-1}).$$

A *deformation* of the represntation ρ is a family of maps $\rho_t : G \to \mathrm{Aut}(V)$, varying continuosly with $t \in [0,\epsilon]$ such that $\rho_0 = \rho$ and ρ_t is a homomorphism for all t.

We claim that the infinitesimal deformations of ρ modulo conjugtaion deformations are given by $H^1(G; \mathrm{End}(V))$, where the action is $ad(\rho)$. Let $\alpha : G \to \mathrm{End}(V)$. Then $\rho_t(g) = exp(t\alpha(g))\rho(g)$. The homomorphism condition says,

$$\begin{aligned}\rho_t(g_1g_2) &= exp(t\alpha(g_1g_2))\rho(g_1g_2)\\ &= \rho_t(g_1)\rho_t(g_2)\\ &= exp(t\alpha(g_1))\rho(g_1)exp(t\alpha(g_2))\rho(g_2)\end{aligned}$$

But now,

$$\rho_t(g_1g_2) = exp(t\alpha(g_1g_2))\rho(g_1g_2) = exp(t\alpha(g_1))^{\rho(g_1)}exp(t\alpha(g_2))\rho(g_1)\rho(g_2)$$

So, if ρ_t is a homomorphism,

$$exp(t\alpha(g_1g_2)) = exp(t\alpha(g_1))^{\rho(g_1)}exp(t\alpha(g_2)),$$

and thus,

$$\alpha(g_1g_2) = \alpha(g_1) + \rho(g_1\alpha(g_2)).$$

That is, α is a crossed homomorphism. The *representation variety*, $\mathrm{Hom}(G, \mathrm{Aut}(V))$ is an algebraic subvariety of

$$\mathrm{Hom}(G, \mathrm{Aut}(V)) \subset [\mathrm{Aut}(V)]^G = \prod_{g\in G} \mathrm{Aut}(V).$$

If $\rho \in \mathrm{Hom}(G, \mathrm{Aut}(V))$, then we have the *Zariski tangent space* at ρ, $T_\rho \mathrm{Hom}(G, \mathrm{Aut}(V))$.

Claim 4.7.2. *The Zariski tangent space of the representation variety is isomorphic to $Z^1(G, End(V))$.*

Suppose $\rho : [0, \epsilon) \to \mathrm{Hom}(G, \mathrm{Aut}(V))$ such that $\rho(0) = \rho_0$. Then by the Leibniz rule,

$$\rho'(g_1g_2) = \rho'(g_1)g_2 + g_1\rho'(g_2).$$

If $\tau_g \in \mathrm{End}(V)$, then we can write this as,

$$\tau_{g_1g_2}g_1g_2 = \tau_{g_1}g_2g_1 + \tau_{g_2}g_1g_2$$

and thus,

$$\tau_{g_1g_2}g_1g_2 = \tau_{g_1}g_2g_1 + \tau_{g_2}g_1g_2$$

and we obtain,

$$\tau_{g_1g_2} = \tau_{g_1} + g_1 * \tau g_2,$$

which is exactly the cocylce condition. So, $Z^1(G; End(V)) = T_\rho \mathrm{Hom}(G, \mathrm{Aut}(V))$. Now, for conjugation, $\rho_t(g) = (g_t\rho g_t^{-1})'_{t=0}$. Let $g' = \tau \in \mathrm{End}(V)$. Then,

$$\rho_0'(g) = \tau\rho(g) - \rho(g)\tau = \tau\rho(g) - (\rho(g) * \tau)\rho(g),$$

and thus,

$$\tau - (\rho(g) * \tau)$$

and this is the trivial crossed homomorphism.

5 Sheaves

We begin with the notion of a presheaf. Let X be a topological space. Then we have the category $Open(X)$ with objects the open sets of X and $\mathrm{Hom}(U, V) = \{i : U \hookrightarrow V\}$ if $U \subset V$ and $\mathrm{Hom}(U, V) = \emptyset$ otherwise. A *presheaf* A is a contravariant functor from the category $Open(X)$ to the category of groups, abelian groups, rings, R-modules, or k-algebras. For simplicity, we will work with presheaves (sheaves) of abelian groups unless explicitly stated otherwise, but the reader should keep in mind how our statements apply to other categories. So, for each open set $U \subset X$ we have an abelian group $A(U)$. Elements of this group are called the *sections* of the presheaf over U. If $U \subset V$, then we have a homomorphism $r_{V,U} : A(V) \to A(U)$. This map is called the *restriction* of the sections over V to the sections over U, and if $U \subset V$ and $s \in A(V)$, we frequently write $s|U$ for $r_{V,U}(s)$. Since we require the presheaf to be a functor, the restriction mappings must satisfy the usual functorial properties, i.e. $r_{U,U} = \mathrm{Id}_{A(U)}$ and when $U \subset V \subset W$ then $r_{V,U} \circ r_{W,V} = r_{W,U}$.

A sheaf is a presheaf which satisfies the following two conditions known as the *sheaf axioms*.

Sheaf Axiom 1
If $U = \bigcup\limits_{\alpha} U_\alpha$ with $U_\alpha \subset X$ open, and $s, t \in A(U)$ are sections over U such that $s|U_\alpha = t|U_\alpha$ for all α then $s = t$.

Sheaf Axiom 2
Let $\{U_\alpha\}$ be a collection of open sets in X and let $U = \bigcup\limits_{\alpha} U_\alpha$. If $s_\alpha \in A(U_\alpha)$ are given so that $s_\alpha|(U_\alpha \cap U_\beta) = s_\beta|(U_\alpha \cap U_\beta)$ for every α and β, then there is a section $s \in A(U)$ with $s|U_\alpha = s_\alpha$ for each α.

The sheaf axioms can be put more consicely by ordering the index set and then saying that the following sequence is exact.

$$0 \longrightarrow A(U) \xrightarrow{\ f\ } \prod_{\alpha} A(U_\alpha) \xrightarrow{\ g\ } \prod_{\alpha<\beta} A(U_\alpha \cap U_\beta)$$

where $U = \bigcup\limits_{\alpha} U_\alpha$ and f is given by $s \mapsto \prod s|U_\alpha$ for $s \in A(U)$, and g is given by $\{s_\alpha\} \mapsto \prod_{\alpha<\beta}\{s_\alpha|U_\alpha \cap U_\beta - s_\beta|U_\alpha \cap U_\beta\}$ for $s_\alpha \in A(U_\alpha)$. It is a simple exercise to check that the exactness of this sequence is equivalent to the two sheaf axioms.

A sheaf on X is said to be a sheaf of groups, abelian groups, rings, R-modules, or k-algebras if the functor describing the underlying presheaf is a functor from the category of open subsets of X to the category of groups, rings, R-modules, or k-algebras. This means that the sections over any open subset of X carries the corresponding algebraic structure and that the restriction mappings are morphisms of these structures. Notice that in all these cases if the sheaf axioms hold for the underlying presheaf of sets making the presheaf a sheaf of sets then the axioms hold in the algebraic category as well and we have a sheaf with the corresponding algebraic structure. This, however, is not always true. For example, this may not be true if the underlying presheaf is a functor to the category of topological spaces.

Exercise 5.0.3. *Define a presheaf in the category of topological spaces and show that a presheaf in this category may determine a sheaf of sets without being a sheaf of topological spaces.*

5.1 Examples of Sheaves

5.1.1 Structure Sheaves

Let X be a topological space. We define a sheaf of $\mathbb{R}$-algebras $C^0(X)$ over X. The sections $C^0(U)$ over an open subset $U \subset X$ are the continuous real-valued functions on U. For an inclusion $V \subset U$ of open subsets of X, the mapping $C^0(U) \to C^0(V)$ is the usual restriction mapping on continuous functions. Clearly, this is a functor: identities go to identities and composition is preserved. Also, since a function is determined by its values, if $\{U_i\}$ is an open covering of U then an element $f \in C^0(U)$ is completely determined by its restrictions $f_i = f|_{U_i}$. Since continuity is a local property, given functions $f_i \in C^0(U_i)$ they patch together to form an element $f \in C^0(U)$ whose restriction to U_i is f_i if and only if for every pair of indices i, j, we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. These are exactly the axioms that are required for $C^0(X)$ to be a sheaf. It is the *structure sheaf of the topological space X*.

Let M be a smooth manifold. We define a sheaf of $\mathbb{R}$-algebras $C^\infty(M)$, the *sheaf of smooth functions* on M as follows. The sections $C^\infty(U)$ over an open subset $U \subset M$ are the smooth functions on U. For an inclusion $V \subset U$ of open subsets of M, the map $r_{U,V} C^\infty(U) \to C^\infty(V)$ is the usual restriction mapping. This clearly defines a functor, i.e., a presheaf. Again, since functions are determined by their values and smoothness is a local condition, it follows that the sheaf axioms hold for this sheaf. It is the structure sheaf of the smooth manifold M. Notice that $C^\infty(M)$ is a subsheaf of $C^0(M)$, which means that for each open subset U we have $C^\infty(U) \subset C^0(U)$ in a manner compatible with the restriction mappings.

Now let M be a real analytic manifold. This means that M is covered by coordinate charts identified with open subsets of $\mathbb{R}^n$ in such a way that the overlap functions are real analytic. On such manifolds we have the notion of a function $f: M \to \mathbb{R}$ being real analytic. This simply means that when restricted to any of the real analytic charts it becomes a real analytic function on the given open subset of $\mathbb{R}^n$. This notion of course makes sense for any open subset of M. In this case, we have the structure sheaf of $\mathbb{R}$-algebras $C^\omega(M)$ of real analytic functions on M. It associates to an open set U the $\mathbb{R}$-algebra of real analytic functions on U, with restriction being the usual restriction mapping. The sheaf axioms are a consequence of the fact that a function is determined by its values and a function on a real analytic manifolds is real analytic if and only if it is real analytic in a neighborhood of every point. When M is a real analytic manifold we have $C^\omega(M) \subset C^\infty(M) \subset C^0(M)$.

Now let M be a complex analytic manifold. Its structure sheaf is the sheaf of $\mathbb{C}$-algebras which associates to each open subset $U \subset M$ the $\mathbb{C}$-algebra of complex-valued, complex analytic functions. The restriction mapping is the usual one. Again the sheaf axioms for this presheaf hold because a function is determined by its values and a function on a complex analytic manifold is complex analytic if an only if it is complex analytic in some neighborhood of each of its points.

5.1.2 Schemes

Now let us give a related, but somewhat different example of a structure sheaf. Let k be a field and let $X \subset k^n$ be an affine algebraic variety defined over k. This means that there is a prime ideal I_X in the polynomial ring $k[x_1, \ldots, x_n]$ consisting of all functions vanishing on X. (Strictly speaking it is the ideal, not the subset that is most important – one gives a variety by giving the ideal.) The coordinate ring $k[X]$ of the variety is the quotient ring $k[x_1, \ldots, x_n]/I_X$. Since we assumed that I_X is a prime ideal, it follows that $k[X]$ is an integral domain. It is the *ring of regular functions* on the variety X.

In this case one works in the Zariski topology rather than a classical topolgy. The space is denoted $\mathrm{Spec}(k[X])$, and is called the *spectrum* of $k[X]$. Its points in the space are the subvarieties of X, or equivalently, the prime ideals in $k[X]$. The topology is defined by specifying the closed subsets. For any point $Y \in \mathrm{Spec}(k[X])$ is a prime ideal $I_Y \subset k[X]$. We consider all prime ideals $I_Z \in \mathrm{Spec}(k[X])$ which contain I_Y. (In terms of subvarieties, this means that $Z \subset Y$.) In any event, the union of all such I_Z containing I_Y is defined to be a closed subset in the Zariski topology, the closed subset of all subvarieties of Y. The general closed subset is a finite union of these basic closed sets associated to subvarieties. It is easy to check that this defines a topology. The maximal ideals are called the closed points of $\mathrm{Spec}(k[X])$. Indeed:

Exercise 5.1.1. *Show that closed points are the only points of* $\mathrm{Spec}(k[X])$ *which are closed subsets in the Zariski topology.*

This indicates a defect of the Zariski topology – it is not Hausdorff, and indeed points are not closed.

Exercise 5.1.2. *Show that any two open sets in the Zariski topology have a non-empty intersection.*

For example if $C \subset \mathbb{C}^2$ is an algebraic curve defined by a single irreducible polynomial equation, then the closed subsets of C are C itself and finite subsets of C. The open sets are then the empty set and complements of finite subsets. In particular, any two non-empty open subsets have non-empty intersection. This is true in general in the Zariski topology.

Now suppose $\mathfrak{p} \in \mathrm{Spec}(k[X])$ is a point. Then we form the localization $k[X]_{(\mathfrak{p})}$ of $k[X]$ at $\mathfrak{p}$. By definition this is the ring obtained by inverting all elements $g \in k[X]$ which do not vanish along the variety $Y_{\mathfrak{p}}$ associated to $\mathfrak{p}$, i.e., do not belong to $\mathfrak{p}$. The elements of this ring are represented by formal fractions f/g where $g \notin \mathfrak{p}$. Two such, f_1/g_1 and f_2/g_2, are identified if $f_1 g_2 = f_2 g_1$ in $k[X]$. One sees easily that $k[X] \subset k[X]_{(\mathfrak{p})}$. More generally, if $Z \subset Y$ are subvarieties (this means that if $\mathfrak{q}$ and $\mathfrak{p}$ are the prime ideals associated to Z and Y, respectively, then we have $\mathfrak{p} \subset \mathfrak{q}$), then $k[X]_{(\mathfrak{q})} \subset k[X]_{(\mathfrak{p})}$. In the special case when $Y = X$, (i.e., $\mathfrak{p} = 0$, this localization is the quotient field $k(X)$ of $k[X]$, and all the intermediate localizations are subrings of $k(X)$ containing $k[X]$. Now we define the k-algebra of sections of the structure sheaf over an open subset $U \subset \mathrm{Spec}(k[X])$. We consider all functions $\varphi\colon U \to \coprod_{\mathfrak{p}\in U} k[X]_{\mathfrak{p}}$ satisfying two properties:

1. For every $\mathfrak{p} \in U$ we have $\varphi(\mathfrak{p}) \in k[X]_{(\mathfrak{p})}$.
2. For every $\mathfrak{p} \in U$ there is an open neigborhood V of $\mathfrak{p}$ in U and elements $f, g \in k[X]$ such that for every $\mathfrak{q} \in V$ we have $g \notin \mathfrak{q}$ and $\varphi(\mathfrak{q}) = [f/g]$ in $k[X]_{(\mathfrak{q})}$.

In words, over U we consider functions of the form f/g, with $f, g \in k[X]$, where g doesn't vanish at any point of U, i.e., g is not contained in any prime ideal which is a point of U. The restriction function is defined in the obvious way. Clearly, this gives us a presheaf of k-algebras over $\mathrm{Spec}(k[X])$. Since this construction defines a section of the presheaf over U in terms of its values at all the points of U, it is clear that the first sheaf axiom holds. Secondly, the set of functions that we consider are also determined locally, that is to say if a function on U satisfies a the property to be a section of the presheaf in a neighborhood of every point of its domain then it is a section of the presheaf over its entire domain. This implies the second sheaf axiom. This is the structure sheaf of this affine variety.

Exercise 5.1.3. *The above definition extends from affine varieties defined over fields to arbitrary commutative rings with unit. Let R be such a ring. Define* $\mathrm{Spec}(R)$ *with its Zariski topology and define the structure sheaf over* $\mathrm{Spec}(R)$ *generalizing the definitions above when $R = k[X]$. These objects, the spectrum of a ring with its Zariski topology and its structure sheaf is called an affine scheme. Describe this data in the case $R = \mathbb{Z}$.*

5.1.3 Pushforward Sheaves

Supppose that $f\colon X \to Y$ be a continuous map between topological spaces and let ξ be a sheaf of abelian groups on X. We define the pushforward $f_!(\xi)$, a sheaf on Y as follows. The sections of $f_!(\xi)$ over an open subset $U \subset Y$ are the sections of ξ over $f^{-1}(U) \subset X$. The restriction mappings are the natural ones. Clearly, $f_!(\xi)$ is a functor from the category of open subsets of X to abelian groups. Since the two sheaf axioms hold for ξ, it is straightforward to see that they also hold for $f_!(\xi)$.

Exercise 5.1.4. *Show that the sheaf axioms hold for $f_!(\xi)$.*

As an example, let $f\colon M \to N$ be a smooth map betwen smooth manifolds. For each $k \geq 0$ we have $f_!\Omega^k(M)$ which assigns to an open subset $U \subset N$ the smooth k-forms on $f^{-1}(U) \subset M$.

5.1.4 Constant Sheaves

Let A be an abelian group and let X be a topological space. The *constant sheaf over X with values in A* is a sheaf of abelian groups whose value over an open subset $U \subset X$ is the set of locally constant functions from U to A, i.e., an assignment of an element of A to each connected component of U. Restriction is the obvious one. Since this is a presheaf of functions, it satisfies the first sheaf axiom. As to the second, a locally constant function is locally constant if and only if it is locally constant in a neighborhood of each point. From this, one easily establishes the second sheaf axiom.

Exercise 5.1.5. *Show that if we replace "locally constant" by "constant" in the above definition then the result is a preshaef that is not a sheaf.*

5.1.5 Locally Constant Sheaves

A sheaf ξ on X is said to be locally constant if in a neighborhood of each point $x \in X$ it is isomorphic to a constant sheaf. That is to say, there is an open covering $\{U_\alpha\}$ of X such that the restriction of ξ to each U_α is isomorphic to a constant sheaf on U_α.
Example: Suppose that $\widetilde{X}$ is a topological space with a free, properly discontinuous action of a discrete group G. Denote the quotient $X = \widetilde{X}/G$. Suppose that G acts as on an abelian group A. Then there is an action of G on the trivial sheaf on $\widetilde{X}$ with values in A covering the given action of G on $\widetilde{X}$. The quotient is a sheaf of X which is locally isomorphic to the trivial sheaf on X with coefficients in A. In fact for any open subset $U \subset X$ which lifts to $\widetilde{X}$, any such lifting determines an isomorphism from the quotient sheaf over U to the original trivial sheaf over the image on U in $\widetilde{X}$.

5.1.6 Sections of Vector Bundles.

Let X be a topological space and $E \to X$ a real vector bundle. Then we have the sheaf of sections of E. This is a sheaf of $\mathbb{R}$-vector spaces. The sections of the sheaf over an open subset U consist of the $\mathbb{R}$-vector space of sections of $E|_U$. Restriction is the obvious one. Again the sheaf axioms are straightforward to establish. Notice that this sheaf is a module over the sheaf of continuous functions in the sense that for each open set $U \subset X$, the sections of $E|_U$ are a module over $C^0(U)$ and these module structures are compatible under restriction.

If M is a smooth manifold and $E \to M$ is a smooth vector bundle, then we have the sheaf of smooth sections of E, which form a sheaf of $\mathbb{R}$-vector spaces which are modules over the sheaf $C^\infty(M)$. Similarly, if M is real or complex analytic and $E \to M$ is a real or complex analytic bundle we have the sheaf of real or complex analytic sections which form modules over the sheaf of real or complex analytic functions on M.

5.2 Basic Constructions with Sheaves

The presheaves of abelian groups over X form a category. A morphism $\varphi\colon F \to G$ is a collection of homomorphisms $\varphi(U)\colon F(U) \to G(U)$ for each open subset $U \subset X$ which are compatible with the restriction mappings of F and G. Clearly there are identity morphisms and composition of morphisms so that we have indeed formed a category. In fact, this is an abelian category. The morphisms from F to G form an abelian group in the obvious way, and the kernel and cokernels of a morphism of presheaves is simply the family of kernels and cokernels of the $\varphi(U)$ as U varies over the open subsets of X.

Inside this category there is the full subcategory of sheaves. The objects of this subcategory are the presheaves satisfying the two sheaf axioms and the morphisms between two sheaves are simply the morphisms between the underlying presheaves. Interestingly, as we shall see below, this subcategory is not abelian since cokernels do not always exist.

Let F be a presheaf of abelian groups over X. We wish to define the sheafification of F. This is a sheaf $\overline{F}$ over X and a morphism of presheaves $F \to \overline{F}$ which is universal for all morphisms of F to sheaves over X. To construct $\overline{F}$ we first construct the étalé space over X associated to F. For each $x \in X$ we define the germ F_x of F at x as $\varinjlim_{x\in U} F(U)$ where the direct limit is taken over the directed set of open neighborhoods of x. This direct limit is an abelian group. The direct limit F_x is called the *stalk* of F at x.

Exercise 5.2.1. *Show that if F is the sheaf of C^∞-functions on a smooth manifold M, then F_x is the germs of C^∞-functions on M at x.*

5.2.1 The Étale Space of a Presheaf

Let F be a presheaf of abelian groups over X. We shall define a space Et_F, the étalé space of F over X. It is the disjoint union over $x \in X$ of the stalks F_x with the topology being defined as follows. Let $U \subset X$ be an open set and let $\alpha \in F(U)$ be a section of the presheaf F over U. Then for each $x \in U$ we have the image $\alpha_x \in F_x$. The collection $\{x, \alpha_x\}_{x\in U}$ is defined to be an open subset of Et_F.

Lemma 5.2.2. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be open sets of this type in Et_F. Then $\mathcal{U}_1 \cap \mathcal{U}_2$ is also an open subset of this type.*

Proof. Let U_1 and U_2 be the images of these open subsets in X and $f_1 \in F(U_1)$ and $f_2 \in F(U_2)$ be the sections defining $\mathcal{U}_1$ and $\mathcal{U}_2$. By definition U_1 and U_2 are open subsets of X. Let $U = U_1 \cap U_2$. Then we have the restrictions $f_1|U$ and $f_2|U$ in $F(U)$. The lemma will follow immediately if we can show the following:

Claim 5.2.3. *The set of $x \in U$ for which $[f_1] = [f_2] \in F_x$ is an open subset.*

Proof. If $[f_1] = [f_2] \in F_x$, then there is an open neighborhood V of x for which $f_1|V = f_2|V$. Hence, $[f_1] = [f_2] \in F_y$ for all $y \in V$. This completes the proof of the claim. □

Now $\mathcal{U}_1 \cap \mathcal{U}_2 = (V, f_1|V)$ where $V = \{x \in U_1 \cap U_2 | [f_1] = [f_2] \in F_x\}$ and by the previous claim V is an open subset of X. □

It follows that these open sets form a basis for a topology on X. This means that a subset of Et_F is open if and only if it is a union of sets of this type. This is the étalé topology on Et_F.

Exercise 5.2.4. *Show that the natural projection mapping* $\mathrm{Et}_F \to X$ *is continous and is a local homemomorphism.*

Exercise 5.2.5. *Show that the addition maps in the stalks leads to a continuous map* $\mathrm{Et}_F \times \mathrm{Et}_F \to \mathrm{Et}_F$ *commuting with the projections to* X*.*

Exercise 5.2.6. Et_F *is not necessarily a Hausdorff space. Give an example for which it is not.*

Now given Et_F, we define the sheaf $\overline{F}$ to be the sheaf of sections of $\mathrm{Et}_F \to X$. That is to say $\overline{F}(U)$ is the abelian group of all continuous maps $U \to \mathrm{Et}_F$ which project to the inclusion of $U \to X$. The restriction mappings are restrictions of sections. The group structure is that induced by addition in the stalks. This addition is compatible with the restriction maps, so that we have defined a presheaf of (local) sections, a presheaf of abelian groups. Since the elements in our sheaves are functions and functions are determined by their values, the first sheaf axiom holds for $\overline{F}$. As for the second, let $U = \cup_i U_i$ and suppose we are given local sections over U_i which agree on the overlaps. We can piece these together to give a function $U \to \mathrm{Et}_F$ which is clearly a continous section whose restriction to each of the U_i is as required. This is the second sheaf axiom, completing the proof that $\overline{F}$ is a sheaf of abelian groups.

Lemma 5.2.7. *There is a natural map of presheaves of abelian groups* $F \to \overline{F}$ *which induces an isomorphisms on the stalks at every point.*

Proof. For any open subset $U \subset X$ we have the natural mapping $F(U) \to \overline{F}(U)$. Since these maps are compatible with the restriction mappings, they determine a map of presheaves of abelian groups. Passing to the direct limits, they induce maps on the stalks $F_x \to \overline{F}_x$. Let U be an open neighborhood of x. Any section $\alpha \in \overline{F}(U)$ has the property that its restriction to some smaller neighborhood U' of x is the image of a section of $F(U')$. This shows that the map on stalks is onto. Conversely, if a section $\alpha \in F(U)$ maps to zero in $\overline{F}_x$, this means that the local section of Et_F determined by α takes value 0 at F_x, and hence the map on stalks is one-to-one. $\square$

Lemma 5.2.8. *If F is a sheaf, then the natural inclusion* $F \to \overline{F}$ *is an isomorphism of sheaves.*

Proof. By the previous lemma, the inclusion $F \to \overline{F}$ induces an isomorphism on the stalks at every $x \in X$. Suppose a section $\alpha \in F(U)$ maps to zero in $\overline{F}(U)$. This means that the image of α in every stalk F_x, $x \in U$, is trivial. That is to say, for each $x \in U$

there is a neighborhood U_x of x such that the restriction $\alpha|U_x = 0$ in $F(U_x)$. Hence, we can cover U by open sets on which α is trivial. By the first sheaf axiom for F, this implies that $\alpha = 0$ in $F(U)$. Now let $\overline{\alpha} \in \overline{F}(U)$. Since the inclusion is an isomorphism on the stalks, we can cover U by open subsets U_i such that $\overline{\alpha}_i = \overline{\alpha}|_{U_i}$ is the image of a section $\alpha_i \in F(U_i)$. Since the $\overline{\alpha}_i$ satisfy the gluing condition on overlaps, and since we have already shown that $F(V) \to \overline{F}(V)$ is an injection for every open subset V of X, it follows that the α_i also satisfy the gluing condition on overlaps. Hence, by the second sheaf axiom for F, there is a section $\alpha \in F(U)$ whose restriction to each U_i is equal to α_i. Then the image of α in $F(U)$ has the same restriction as $\overline{\alpha}$ to each U_i and hence by the first sheaf axiom for $\overline{F}$, the image of α is equal to $\overline{\alpha}$. □

Exercise 5.2.9. *Show that if F is a presheaf of rings, R-modules or k-algebras, then the same is true for $\overline{F}$.*

Exercise 5.2.10. *Show that $\overline{F}$ has the stated universal property for maps of F to sheaves over X.*

Exercise 5.2.11. *Show that this construction satisfies a universal property: If $\varphi\colon F \to G$ is a map from F to a sheaf G of abelian groups then φ factors uniquely as the composition of the natural map $F \to \overline{F}$ followed by a map of sheaves $\overline{F} \to G$.*

5.2.2 Pullbacks of Sheaves

Let $f\colon X \to Y$ be a continous mapping and suppose that ξ is a sheaf of abelian groups on Y. We wish to define the pullback $f^*\xi$, a sheaf of abelian groups on X. Let $E(\xi) \to Y$ be the étalé space of ξ. We form the fibered product

$$\begin{array}{ccc} f^*(E(\xi) & \longrightarrow & E(\xi) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Then the sheaf $f^*\xi$ is defined to be the sheaf of sections of $f^*(E(\xi)) \to X$.

Exercise 5.2.12. *Show that there is a natural map $\xi \to f_!(f^*(\xi))$.*

The pullback is the universal solution to this mapping question. Unlike the pushforward operation that can be performed for sheaves in any abelian category, the pullback operation requires the étalé space, and hence is only valid in a category of sheaves whose values are sets with extra structure (such as abelian groups, modules over a ring, etc.) The pullback preserves compositions and sends identities to identities. Thus, pullback makes the category of sheaves of abelian groups over topological spaces into a category. The objects are pairs (X, ξ) consisting of a topological space and a sheaf of abelian groups over it. A morphism from (X, ξ) to (Y, ζ) is a continuous mapping $f\colon X \to Y$ and a homomorphism of sheaves over X $\phi\colon \xi \to f^*\zeta$.

5.3 Kernels and Cokernels

Let F and G be sheaves of abelian groups over a topological space X, and let $\varphi\colon F \to G$ be a morphism. Define the kernel of φ, $\mathrm{Ker}(\varphi)$, to be the presheaf $\mathrm{Ker}(\varphi)(U) = \mathrm{Ker}(\varphi(U)\colon F(U) \to G(U))$. The restriction mappings are defined using the restriction mappings for the sheaf F. Clearly, this is a presheaf and is a subpresheaf of F.

Lemma 5.3.1. $\mathrm{Ker}(\varphi)$ *is a sheaf and its stalk at $x \in X$ is the kernel of the map $\varphi_x\colon F_x \to G_x$ induced by φ.*

Proof. Let U be an open subset of X and $U = \cup_i U_i$. Suppose that $\alpha \in \mathrm{Ker}(\varphi)(U)$ and that $\alpha|_{U_i} = 0$ for all i. Then by the first sheaf property for F, it follows that $\alpha = 0$. Conversely, suppose given $\alpha_i \in \mathrm{Ker}(U_i)$ satisfying the gluing property. By the second sheaf axiom for F, there is $\alpha \in F(U)$ such that for all i the restriction of α to U_i is α_i. Since $\varphi(\alpha_i) = 0$, it follows that $\varphi(\alpha)|_{U_i} = 0$ for all i. Hence, by the first sheaf axiom for G, we see that $\varphi(\alpha) = 0$, and hence that $\alpha \in \mathrm{Ker}(\varphi)(U)$.

Since kernels commute with direct limits, it follows that the stalk of $\mathrm{Ker}(\varphi)$ at x agrees with the kernel of the map $\varphi_x\colon F_x \to G_x$. □

It is clear that if $\psi\colon F' \to F$ is a map of sheaves with the property that $\varphi \circ \psi = 0$, then ψ factors through the inclusion $\mathrm{Ker}(\varphi) \to F$.

Now let us consider the presheaf which is the cokernel of φ. This is a presheaf whose sections over U are $G(U)/\varphi(F(U))$. Unfortunately, this presheaf is not usually a sheaf. We define $\mathrm{Coker}(\varphi)$ to be the sheaf obtained from this presheaf. Direct limits also commute with taking cokernels, so that the stalk of this presheaf at any $x \in X$ is the cokernel of $\varphi_x\colon F_x \to G_x$. It then follows from the property above that:

Lemma 5.3.2. *For every $x \in X$, there is an exact sequence of stalks*

$$0 \to \mathrm{Ker}(\varphi)_x \to F_x \to G_x \to \mathrm{Coker}(\varphi)_x \to 0.$$

If $\psi\colon G \to G'$ is any morphism of sheaves of abelian groups, and if $\psi \circ \varphi = 0$, then ψ factors through the natural map of G to $\mathrm{Coker}(\varphi)$.

This definition turns out to be a reasonable definition in the category of sheaves of abelian groups over X, and makes that category into an abelian category. In particular, a sequence of sheaves

$$\cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} F_3 \longrightarrow \cdots$$

is exact at F_2 if the natural map of sheaves $\mathrm{Coker}(\varphi_1) \to \mathrm{Ker}(\varphi_2)$ is an isomorphism.

5.4 Čech Cohomology with Values in a Sheaf

Let ξ be a sheaf of abelian groups over a topological space X.

Lemma 5.4.1. *The sections of ξ over the empty set are the zero group.*

Proof. From the construction of any sheaf as the sheaf of sections of an étalé space over X, the set of sections over the empty set consists of a single element. Thus, it is the trivial group. $\square$

Given an open covering $\{U_\alpha\}_\alpha$ of X we define the Čech cochains of ξ with respect to this open cover as follows: First choose a total ordering on the index set and then define

$$\check{C}^k = \prod_{\alpha_0 < \cdots < \alpha_k} \xi(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

Of course, by the above lemma, it suffices to take the sum over those multi-indices whose associated intersection is non-empty. The coboundary map is defined as follows: If $a \in \check{C}^k$, then for any $\alpha_0 < \cdots < \alpha_{k+1}$ we have

$$\delta(a)(\alpha_0, \ldots, \alpha_{k+1}) = \sum_{j=0}^{k+1} (-1)^j r^* a(\alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{k+2}),$$

where r^* denotes the appropriate restriction mapping in the definition of the sheaf. The usual computation shows that $\delta^2 = 0$ and hence that we have a complex of abelian groups. The cohomology of this complex is the Čech cohomology with values in ξ with respect to this open covering.

Exercise 5.4.2. *Use arguments as in the case of constant coefficents show that under a refinement there is a well-defined map on cohomology.*

The Čech cohomology with values in ξ is then the direct limit over all open coverings of the Čech cohomology of ξ with respect to the open covering. It is denoted $\check{H}^*(X;\xi)$.

For suppose that $(f,\phi)(X,\xi) \to (Y,\zeta)$ is a morphism in the category of sheaves of abelian groups over topological spaces. There is a natural mapping Prove this.

$$(f,\phi)^* \colon \check{H}^*(Y;\zeta) \to \check{H}^*(X;\xi).$$

This makes Čech cohomology a functor from the category of sheaves of abelian groups over topological spaces to the category of graded abelian groups.

We are going to approach things differently. Instead of using Čech cohomology of a sheaf, we shall give the derived functor approach to the cohomology of sheaves.

5.5 Derived Functors of H^0

5.5.1 Lemmas about exact sequences and groups of sections

In order to prepare the way for the derived functor approach to sheaf cohomology, we need to study how exact sequences of sheaves behave under the operation of taking sections over a given open subset.

Lemma 5.5.1. *Suppose that we have an exact sequence of sheaves of abelian groups over X:*

$$0 \to \xi' \to \xi \to \xi''$$

Then for any open subset $U \subset X$ we have an exact sequence:

$$0 \to \xi'(U) \to \xi(U) \to \xi''(U).$$

Proof. Restricting the sheaves to an open subset of X produces an exact sequence of sheaves over that subset. Thus, it suffices to consider the case $U = X$. Suppose that $\sigma \in \xi'(X)$ and the image of σ in $\xi(X) = 0$. This implies that the value of $\sigma_x \in \xi'_X$ in each stalk maps to zero in ξ_x. Since the maps on the levels of stalks are injective, this implies that $\sigma_x = 0$ for all x and hence that $\sigma = 0$. Since the compositions $\xi'_x \to \xi_x \to \xi''_x$ are zero for all x, it is clear that the composition $\xi'(X) \to \xi(X) \to \xi''(X)$ is zero. Lastly, suppose that $\tau \in \xi(X)$ maps to zero in $\xi''(X)$. Then the images $\tau_x \in \xi_x$ map to zero in ξ''_x. Hence for each $x \in X$ there is a neighborhood $U \subset X$ and a lifting $\sigma_U \in \xi'(U)$ of $\tau|_U$. On the overlaps $U \cap U'$ the restrictions of σ_U and $\sigma_{U'}$ both map to $\tau_{U\cap U'}$. But we have already seen that the map $\xi'(U \cap U') \to \xi(U \cap U')$ is an injection. This means that σ_U and $\sigma_{U'}$ agree on the overlap and hence determine a global section $\sigma \in \xi'(X)$ which maps to τ. □

Suppose that $\xi \to \xi'' \to 0$ is exact. It is not true in general that $\xi(X) \to \xi''(X)$ is onto. Let us examine this question in more detail. Given an element $\mu \in \xi''(X)$ using the fact that the maps $\xi_x \to \xi''_x$ are onto, we can find an open covering $\{U\}$ of X and elements $\tau_U \in \xi(U)$ mapping to $\mu|_U$. On the overlap $U \cap U'$, the restrictions of τ_U and $\tau_{U'}$ have the same image in $\xi''(U \cap U')$, namely $\mu|_{U\cap U'}$. Thus, for every $x \in U \cap U'$ there is an element $\sigma_x \in \xi'_x$ which maps to the difference $(\tau_U)_x - (\tau_{U'})_x$. As above, using the injectivity of $\xi' \to \xi$ we see that these elements σ_x glue together to form an element $\sigma_{U,U'} \in \xi'(U \cap U')$ whose image in $\xi(U \cap U')$ is $(\tau_U)|_{U\cap U'} - (\tau_{U'})|_{U\cap U'}$. These then are a Čech one cochain with values in ξ'. If we consider the restriction to triple overlaps we see that the restriction to $U \cap U' \cap U''$ of $\sigma_{U,U'} - \sigma_{(U,U'')} + \sigma_{U',U''}$ maps to zero in $\xi(U \cap U' \cap U'')$. Again using the injectivity of $\xi' \to \xi$ on sections, we see that this implies that the Čech one cochain determined by the $\sigma_{U,U'}$ is a one cocylce. If this cocycle is a coboundary then one can modify the $\sigma_{U,U'}$ by the coboundary of s_U until it becomes zero. This would allow us to modify the τ_U by the image of the s_U so that they agree on the overlap and hence form a global section of ξ mapping to μ. This indicates that the obstruction to the surjectivity of $\xi(X) \to \xi''(X)$ lies in the first Čech cohomology of X with values in ξ'.

This argument has one consequence which is extremely important.

Lemma 5.5.2. *Let us suppose that*

$$0 \to \xi' \to \xi \to \xi'' \to 0$$

is an exact sequence of sheaves of abelian groups over X. Suppose that for every open subset $U \subset X$ the restriction mapping $\xi'(X) \to \xi'(U)$ is surjective. Then for every open subset U, $\xi(U) \to \xi''(U)$ is surjective.

Proof. Again it suffices to consider the case when $U = X$. Let $t \in \xi''(X)$. Consider pairs (U, s) consisting of an open subset of X and an element $s \in \xi(U)$ mapping to t. These pairs are naturally ordered: $(U, s) < (U', s')$ if $U \subset U'$ and $s = s'|_U$. Any totally ordered chain has a maximal element – take the union of the open subsets and use the sheaf axiom to glue together the sections to form a section on the union. Thus, by Zorn's lemma there is maximal element (U, s). Suppose that $U \neq X$. Then there is a point $x \notin U$. There is a local section s' defined in some neighborhood V of x which maps to $t|_V$. Consider $s|_{U\cap V} - s'|_{U\cap V}$. This section extends to a section s'' on V. Clearly, $(s' + s'')|_{U\cap V} = s|_{U\cap V}$ and $s' + s''$ maps to $t|_V$. Using the sheaf axioms we glues $(s' + s'') \in \xi'(V)$ and $s \in \xi'(U)$ to form $\hat{s} \in \xi'(U \cup V)$. Clearly $\hat{s}$ maps to t. So, we have a section over $U \cup V$ extending s on U and mapping to t, contradicting the maximality of (U, s). This contradiction implies that $U = X$ and completes the proof. □

Sheaves with the property stated in the lemma are called *flabby* (*flasque* in French).

5.5.2 The derived functor construction

Let ξ be a sheaf of abelian groups over X. We define $H^0(X, \xi) = \xi(X)$, the group of global sections of ξ. This is a functor from the category of sheaves of abelian groups over X to the category of abelian groups. The higher cohomology groups of ξ are defined as the (right) derived functors of this functor. By this we mean we have a functor $H^*(X; \xi)$ from the category of sheaves to the category of graded abelian groups (with gradings in degrees ≥ 0) such that H^0 is the global section functor and such that associated to any short exact sequence

$$0 \to \xi' \to \xi \to \xi'' \to 0$$

we have a long exact sequence of cohomolgy groups:

$$\cdots \to H^i(X; \xi') \to H^i(X; \xi) \to H^i(X; \xi'') \to H^{i+1}(X; \xi') \to \cdots$$

functorial in maps between short exact sequences. It may not be clear from this description that this completely determines the higher cohomology groups, but that is exactly what the homological algebra of derived functors allows one to prove. We shall not show this general uniqueness statement. Rather, we will give an explicit construction and show that it satisfies these axioms, and then define that as the cohomology functor.

Let ξ be a sheaf of abelian groups over X. The construction we give is based on the étalé space $\acute{E}t(\xi)$ over X. We define $C(\xi)$ to be the presheaf which assigns to an open subset $U \subset X$ the group of all sections, continuous or not, $U \to \acute{E}t(\xi)|_U$. In other words, $C(\xi)(U) = \prod_{x \in U} \xi_x$. The restriction mapping is the obvious one as is the inclusion of $\xi \to C(\xi)$.

Exercise 5.5.3. *Check that $C(\xi)$ is a sheaf.*

Furthermore, $C(\xi)$ is clearly flabby – given a section over U we can just extend it to be zero outside of U (since there is no requirement of continuity).

Now we define $C^0(\xi) = C(\xi)$. We have the natural inclusion $\xi \hookrightarrow C^0(\xi)$. Let $B^0(\xi)$ be the cokernel of this map of sheaves, and define $C^1(\xi) = C(B^0(\xi))$. We have the natural composition $C^0(\xi) \to B^0(\xi) \subset C^1(\xi)$. Suppose inductively that we have defined $C^i(\xi)$ for all $i < n$ together with maps $C^i(\xi) \to C^{i+1}(\xi)$. Then we let $B^{n-1}(\xi)$ be the cokernel of $C^{n-2}(\xi) \to C^{n-1}(\xi)$ and we define $C^n(\xi) = C(B^{n-1}(\xi))$. Continuing in this way we define the $C^n(\xi)$ for all $n \geq 0$ and maps $C^n(\xi) \to C^{n+1}(\xi)$.

Lemma 5.5.4. *The $C^n(\xi)$ are all flabby for $n \geq 0$ and we have an exact sequence of sheaves:*

$$0 \to \xi \to C^0(\xi) \to C^1(\xi) \to \cdots$$

Proof. Since for any sheaf of abelian groups ζ the sheaf $C(\zeta)$ is flabby, it follows immediately from the construction that the $C^n(\xi)$ are all flabby. It is also immediate from the constrution that the composition $C^n(\xi) \to C^{n+1}(\xi) \to C^{n+2}(\xi)$ is zero, as well as the composition $\xi \to C^0(\xi) \to C^1(\xi)$. The last thing to check is exactness. Suppose that an element $a_x \in C^n(\xi)_x$ maps to zero in $C^{n+1}(\xi)_x$. Then it maps to zero in $B^n(\xi)_x$, which means that it is in the image of $C^{n-1}(\xi)_x$. This completes the proof of exactness. □

Since we have a chain complex $C^0(\xi) \to C^1(\xi) \to \cdots$, taking global sections

$$H^0(X; C^0(\xi)) \to H^0(C^1(X;\xi) \to \cdots$$

leads to a complex of abelian groups. The cohomology of this complex is defined to be the cohomology of ξ.

Lemma 5.5.5. *With the above definition, the cohomology groups are a functor from the category of sheaves of abelian groups over X to the category of graded abelian groups with non-trivial groups only in non-negative degrees. Furthermore, $H^0(X;\xi)$ is identified with the global sections of ξ.*

Proof. Suppose that $\xi \to \zeta$ is a morphism of sheaves of abelian groups over X. Then there is a map of the associated étalé spaces and hence a map $C(\xi) \to C(\zeta)$ compatible with the map $\xi \to \zeta$. Applying the construction of the complex of sheaves, one concludes that that there is an induced map between the complex of sheaves, and hence between the complexes of global sections. This induces then a map on the cohomology groups. From the exact sequence $0 \to \xi \to C^0(\xi) \to C^1(\xi)$ we see that the kernel of the map $H^0(X; C^0(\xi)) \to H^0(X : C^1(\xi))$ is identified with $H^0(X;\xi)$, and hence the zeroth cohomology of the cochain complex is identified with the group of global sections of ξ. This identification is clearly natural for morphisms of sheaves. □

The last thing to establish is the cohomology long exact sequence associated to a short exact sequence of sheaves. Finish Argument

6 Spectral Sequences

Recall that a composition series of finite length for an abelian group A is the sequence of quotient groups A^r/A^{r+1} of a decreasing filtration of finite length

$$A = A^k \supset A^{k+1} \supset A^{k+2} \supset \cdots \supset A^\ell = 0$$

The basic setup for a spectral sequence is a cochain complex (C^*, d) of abelian groups and a decreasing filtration $F^*(C^*)$, which means for each cochain group C^n we have a decreasing sequence of subgroups

$$\cdots F^k(C^n) \supset F^{k+1}(C^n) \supset F^{k+2}(C^n) \supset \cdots$$

This filtration is required to be compatible with the coboundary d in the sense that $d(F^k(C^n)) \subset F^k(C^{n+1})$. We define the associated graded groups for C^n by

$$\mathrm{Gr}^p_{F^*}(C^n) = F^p(C^n)/F^{p+1}(C^n).$$

There is an induced decreasing filtration on cohomology, $H^* = H^*(C^*)$ denoted $F^*(H^*)$. By definition $F^k(H^n)$ consists of all cohomology classes in degree n that have cocycle representatives contained in $F^k(C^n)$. Said another way we have the subcomplex $F^k(C^*)$ of C^* and we define $F^k(H^*)$ to be the image of the cohomology of the subcomplex in $H^*(C^*)$. Clearly, this is a decreasing filtration. We denote by

$$\mathrm{Gr}^*_{F^*}(H^*)$$

the associated graded of this filtration.

To get anything reasonable we need to make some finiteness assumptions. While it is possible to get by with less, we make fairly strong assumptions, which nevertheless are the most common ones encountered in the interesting examples. First, we assume that C^* is bounded below, i.e., that $C^k = 0$ for all k sufficiently small, often $k < 0$ in practice. Next we assume that $F^*(C^*)$ is bounded below in the sense that $F^k(C^*) = C^*$ for all k sufficiently small, again often $k < 0$ in practice. Lastly, we assume that $F^k(C^n) = 0$ for all k sufficiently large, how large depending on n. This condition is called *locally bounded above*.

We define

$$E_0^{p,n-p} = \mathrm{Gr}^p_{F^*}(C^n) = F^p(C^n)/F^{p+1}(C^n).$$

In a similar vein we define

$$E_\infty^{p,n-p} = \mathrm{Gr}^p_{F^*}(H^n) = F^p(H^n)/F^{p+1}(H^n).$$

Fixing n and varying p,the groups $E_0^{p,n-p}$ give a composition series of finite length for C^n. Similarly, the groups $E_\infty^{p,n-p}$ give a composition series of finite length for H^n. The idea is to interpolate between these groups by defining a sequence of groups $E_r^{p,n-p}$ for $r = 0, 1, \ldots,$, such that $E_r^{p,q} = E_\infty^{p,q}$ for all r sufficiently large, and differentials $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $E_{r+1}^{p,q}$ is the cohomology of d_r at $E_r^{p,q}$. Thus, we begin with a composition series for $F^*(C^*)$, repeatedly take cohomology and arrive at the composition series for $F^*(H^*)$. Of course, this is a very complicated proceedure, with many groups and differentials to compute, and one can well wonder how it is better than just computing the cohomology directly. The point is that spectral sequences are most useful when the filtration is such that one can identify the early terms in the spectral sequence, usually the E_1- or E_2-terms, with other known cohomology groups. This then gives a first approximation to the final cohomology. Sometimes things are so fortunate that no further computation is even necessary!

Let us now try to understand intuitively how the better and better approximations to the cohomology $E_r^{p,n-p}$ are obtained. Let us begin with $F^p(C^n)$. At stage r we consider classes which are cocycles modulo F^{p+r}, i.e., $x \in F^p(C^n)$ such that $dx \in F^{p+r}(C^{n+1})$ and we divide out by boundaries from F^{p-r+1} as well as all classes in F^{p+1}. Clearly, as r increases we are putting more and more stringent conditions on dx and hence getting closer and closer to the cocycle condition. Eventually, because of the finiteness conditions, for sufficiently large r, we are requiring $dx = 0$. Also, as r increases we are dividing out by more and more coboundaries, and again by the finiteness conditions, for r sufficiently large we are dividing out by all coboundaries. Since we also divide out by F^{p+1} we end up with the associated graded for the filtration on H^n.

Now let me make all this precise.

We define:

$$E_r^{p,q} = \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+r}(C^{p+q+1})\}}{\{x \in F^{p+1}(C^{p+q}) \big| dx \in F^{p+r}(C^{p+q+1})\} + d(F^{p-r+1}(C^{p+q-1})) \cap F^p(C^{p+q})\}}.$$

The map

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

is defined by

$$d_r[x] = [dx].$$

Let us show that this makes sense. Notice that if $[x] \in E_r^{p,q}$ then $x \in F^p(C^{p+q})$ and $dx \in F^{p+r}(C^{p+q+1})$. Of course, $d(dx) = 0 \in F^{p+2r}(C^{p+q+1})$. This means that dx represents a class in $E_r^{p+r,q-r+1}$. If, in addition, $x \in F^{p+1}$ then $dx \in dF^{p+r-1}(C^{p+q}) \cap F^{p+r}(C^{p+q+1})$ and hence the class of dx is trivial in $E_r^{p+r,q-r+1}$. Also if $x \in d(F^{p-r+1}(C^{p+q-1})) \cap F^p(C^{p+q})$ then $dx = 0$ so that dx is trivial in $E_r^{p+r,q-r+1}$. This shows that d induces a well defined mapping d_r as claimed. Since $d^2 = 0$, it is clear that $d_r^2 = 0$ so that for each r we have a bigraded cochain complex $\{E_r^{p,q}, d_r\}$.

Claim 6.0.6. *The cohomology of d_r at $E_r^{p,q}$ is naturally identified with $E_{r+1}^{p,q}$.*

Proof.

Claim 6.0.7.

$$\operatorname{Ker}(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) =$$
$$\frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+r+1}(C^{p+q+1})\}}{\{y \in F^{p+1}(C^{p+q}) \big| dy \in F^{p+r+1}(C^{p+q+1})\} + dF^{p-r+1}(C^{p+q-1}) \cap F^p(C^{p+q})}.$$

Proof. Suppose that $x \in F^p(C^{p,q})$ represents a class $[x] \in E_r^{p,q}$ for which $d_r[x] = 0$. This means that $dx \in F^{p+r}(C^{p+q+1})$ and that

$$dx \in \Big(\{y \in F^{p+r+1}(C^{p+q+1}) \big| dy \in F^{p+2r}(C^{p+q+2}\} + d(F^{p+1}(C^{p+q}) \cap F^{p+r}(C^{p+q+1})\Big).$$

Varying x by an element $y \in F^{p+1}(C^{p+q})$ with $dy \in F^{p+r}(C^{p+q+1})$ does not change $[x] \in E_r^{p,q}$ and allows us to assume that $dx \in \{y \in F^{p+r+1}(C^{p+q+1}) \big| dy \in F^{p+2r}(C^{p+q+2})\}$, which simply means that $dx \in F^{p+r+1}(C^{p+q+1})$.

Now suppose that we have two elements $x, x' \in F^p(C^{p+q})$ representing the same class in $E_r^{p,q}$ satisfy $dx, dx' \in F^{p+r+1}(C^{p+q+1})$. Then their difference y is an element of

$$\{x \in F^{p+1}(C^{p+q}) \big| dx \in F^{p+r}(C^{p+q+1})\} + d(F^{p-r+1}(C^{p+q-1})) \cap F^p(C^{p+q})$$

and $dy \in F^{p+r+1}(C^{p+q+1})$. It follows that

$$y \in \Big(\{x \in F^{p+1}(C^{p+q}) \big| dx \in F^{p+r+1}(C^{p+q+1})\} + d(F^{p-r+1}(C^{p+q-1})) \cap F^{p+1}(C^{p+q})\Big).$$

This completes the proof of the claim. □

Now consider the image of $d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$. Any element in this image is represented by an element of $dF^{p-r}(C^{p+q-1}) \cap F^p(C^{p+q})$. Conversely, any element of this intersection represents an element of $E_r^{p,q}$ that is in the image of d_r. Thus,

$$\begin{aligned} \operatorname{Ker} d_r / \mathrm{Im} d_r &= \operatorname{Ker} d_r : E_r^{p,q} \to E_r^{p+r,q-r+1} \\ &= \frac{\{x \in F^p(C^{p+q}) \big| dx \in F^{p+r+1}(C^{p+q+1})\}}{\{y \in F^{p+1}(C^{p+q}) \big| dy \in F^{p+r+1}(C^{p+q+1})\} + dF^{p-r}(C^{p+q-1}) \cap F^p(C^{p+q})}. \end{aligned}$$

This is exactly the definition of $E_{r+1}^{p,q}$. □

6.1 Double Complexes

One important example of a spectral sequence is the spectral sequence associated to a double complex. A *double complex* consists of a collection of doubly indexed groups $C^{p,q}$ and two anticommuting differntials, $d : C^{p,q} \to C^{p+1,q}$ and $\delta : C^{p,q} \to C^{p,q+1}$ with $d^2 = \delta^2 = 0$, and $d\delta = -\delta d$. Given a double complex we can form the associated *total complex*, $\hat{C}^n = \bigoplus_{p+q=n} C^{p,q}$, with the differential $D : \hat{C}^n \to \hat{C}^{n+1}$ given by $D = d + \delta$. Then $D^2 = 0$:

$$D^2 = (d + \delta)^2 = d^2 + d\delta + \delta d + \delta^2$$

since $d^2 = \delta^2 = d\delta + \delta d = 0$. So, $(\hat{C}^*, D)$ is a cochain complex.

There are two natural filtrations on the total complex of a double complex. First, we can *filter by* p. We define $F^k(\hat{C}^*) = \bigoplus_{\substack{p\geq k \\ p+q=n}} C^{p,q}$. So, $F^0(\hat{C}^*) = \hat{C}^*$ and $F^{n+1}(\hat{C}^n) = 0$. Since D takes $C^{p,q}$ to $C^{p+1,q} \oplus C^{p,q+1}$, the filtration is compatible with the differential, i.e. $D(F^k) \subset F^k$. Using this filtration, the E_0 term of the associated spectral sequence is,

$$E_0^{p,q} = \frac{F^p(\hat{C}^{p+q})}{F^{p+1}(\hat{C}^{p+q})} = C^{p,q}$$

Since,

$$\hat{C}^{p+q} = \bigoplus_{p'+q'=p+q} C^{p',q'}$$

$$F^p(\hat{C}^{p+q}) = \bigoplus_{\substack{p'\geq p \\ p'+q'=p+q}} C^{p',q'}$$

$$F^{p+1}(\hat{C}^{p+q}) = \bigoplus_{\substack{p'\geq p+1 \\ p'+q'=p+q}} C^{p',q'}$$

Furthermore, $d_0 = \delta$. Thus, $E_1^{p,q} = H^q(C^{p,*}, \delta)$. So, there is a spectral sequence with $E_1^{p,q} = H^q(C^{p,*}, \delta)$ converging to $H^*(\hat{C}^*, D)$, the cohomology of the total complex.

We could just have well filtered by q, so lets reverse the roles of $p$and q and see what happens. Now our filtration is given by $(\tilde{F}^k)(\hat{C}^n) = \bigoplus_{\substack{q\geq k \\ p=q}} C^{p,q}$. Similarly, this filtration is compatible with the differential, $D(\tilde{F}^k) \subset \tilde{F}^k$. The spectral sequence in this case has $\tilde{E}_0^{p,q} = C^{p,q}$ and $\tilde{d}_0 = d$ and $\tilde{E}_1^{p,q} = H^p(C^{*,q}, d)$. This spectral sequence also converges to the cohomolgy of the total complex, $H^*(\hat{C}^*, D)$.

Theorem 6.1.1. *Let ξ be a sheaf of abelian groups over X, and suppose*

$$0 \longrightarrow \xi \xrightarrow{\ i\ } \underline{\underline{R}}^0 \xrightarrow{\ f_0\ } \underline{\underline{R}}^1 \xrightarrow{\ f_1\ } \underline{\underline{R}}^2 \xrightarrow{\ f_2\ } \cdots$$

is a resolution. Then there is a spectral sequence whose E_1 term is given by $E_1^{p,q} = H^q(\underline{\underline{R}}^p)$ converging to $H^*(\xi)$, and d_1 is the map induced by the f_i.

We have the following situaition:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^2(\xi) & \longrightarrow & C^2(\underline{\underline{R}}^0) & \longrightarrow & C^2(\underline{\underline{R}}^1) & \longrightarrow & \cdots \\
 & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
0 & \longrightarrow & C^1(\xi) & \longrightarrow & C^1(\underline{\underline{R}}^0) & \longrightarrow & C^1(\underline{\underline{R}}^1) & \longrightarrow & \cdots \\
 & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
0 & \longrightarrow & \xi & \xrightarrow{i} & \underline{\underline{R}}^0 & \xrightarrow{f_0} & \underline{\underline{R}}^1 & \xrightarrow{f_1} & \cdots
\end{array}$$

Where each column is the standard flabby resolution for the sheaf in the bottom row.

Lemma 6.1.2. *If $\xi' \to \xi \to \xi''$ is exact then $C(\xi') \to C(\xi) \to C(\xi'')$is exact at $C(\xi)$.*

Now, we define a double complex by $C^{p,q} = H^0(C^q(\underline{\underline{R}}^p))$, the global sections, and $D = d + \delta$. If we filter on p, taking vertical (δ) cohomology we have $E_1^{p,q} = H^q(\underline{\underline{R}}^p)$ and the spectral sequence converges to $H^*(\hat{C}^*, D)$. If we filter on q, and take horizontal (d) cohomology. Since,

$$0 \longrightarrow C^i(\xi) \longrightarrow C^i(\underline{\underline{R}}^0) \longrightarrow C^i(\underline{\underline{R}}^1) \longrightarrow \cdots$$

is an exact sequence of flabby sheaves, the sequence of global sections H^0 is also exact,

$$H^*(H^0(C^i(\underline{\underline{R}}^*), d)) = \begin{cases} 0 & * > 0 \\ H^0(C^i(\xi)) & * = 0 \end{cases}$$

So we have

$$\tilde{E}_1^{p,q} = \begin{cases} 0 & p > 0 \\ H^0(C^q(\xi)) & p = 0 \end{cases}$$

and,

$$\tilde{E}_2^{p,q} = \begin{cases} 0 & p > 0 \\ H^q(\xi) & p = 0 \end{cases}$$

But all the higher d's are zero, so the sequenc collapses at $\tilde{E}_2$ and we have $H^*(\hat{C}, D) = H^*(\xi)$.

Corollary 6.1.3. *Suppose we have a resolution of ξ*

$$0 \longrightarrow \xi \longrightarrow \underline{\underline{R}}^0 \xrightarrow{f_0} \underline{\underline{R}}^1 \xrightarrow{f_1} \underline{\underline{R}}^2 \longrightarrow \cdots$$

and suppose that $H^i(\underline{\underline{R}}^j) = 0$ for all $i > 0$ and for all $j \geq 0$, then

$$H^*\Big(H^0(\underline{\underline{R}}^0) \to H^0(\underline{\underline{R}}^1) \to H^0(\underline{\underline{R}}^2) \to \cdots\Big) = E_2 = E_\infty = H^*(\xi).$$

6.1.1 Soft sheaves and the relationship between deRham and sheaf cohomology

Now, let M be a smooth manifold. Let $\underline{\underline{\mathbb{R}}}$ be the constant sheaf on M such that $\underline{\underline{\mathbb{R}}}(U) = \mathbb{R}^{comp(U)}$. We want to prove:

Theorem 6.1.4. *The sheaf cohomology of M with coefficents in the constant sheaf $\underline{\underline{\mathbb{R}}}$ is identified with the deRham cohomology, i.e.*

$$H^*(M;\underline{\underline{\mathbb{R}}}) = H^*_{dR}(M;\mathbb{R}).$$

The idea will be as follows. First, by the Poincaré lemma, we have a resolution of the sheaf $\underline{\underline{\mathbb{R}}}$,

$$0 \longrightarrow \underline{\underline{\mathbb{R}}} \longrightarrow \underline{\underline{\Omega}}^0 \xrightarrow{d} \underline{\underline{\Omega}}^1 \xrightarrow{d} \underline{\underline{\Omega}}^2 \xrightarrow{d} \cdots$$

We will prove that the higher cohomologies vanish,

Theorem 6.1.5. $H^i(\underline{\underline{\Omega}}^j) = 0$ *for all* $i > 0$ *and all* $j \geq 0$.

and then the double complex spectral sequence will imply:

$$H^*(M;\underline{\underline{\mathbb{R}}}) = H^*\left(\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots\right) = H^*_{dR}(M;\mathbb{R})$$

To do this we will show that the sheaves $\underline{\underline{\Omega}}^*$ are a special type of sheaves called soft sheaves. Then we wil show that sheaves of this type have some properties very similar to the properties of flabby sheaves. In particular, we will show that the higher cohomologies vanish, and thus prove theorem 6.1.5 above.

Definition 6.1.6. A sheaf ξ is *soft* if every section on a closed set extends to a global section.

By section of a sheaf ξ over a closed set $F \subset X$ we mean a continuous map $\sigma : F \to \mathrm{Et}_\xi$ so that $\pi \circ \sigma = i$, where $\pi : \mathrm{Et}_\xi \to X$ is the standard projction and $i : F \hookrightarrow X$ is the inclusion.

Now we show that the sheaves $\underline{\underline{\Omega}}^*$ are soft. First, we notice that if ξ is a sheaf over a manifold M, and σ_F is a section of this sheaf over a closed set $F \subset M$, then there exists an open neighborhood U of F and a section $\sigma_U \in \xi(U)$ such that $\sigma_U|_F = \sigma_F$. This follows from the fact that $\xi(F) = \varinjlim_{U \supset F} (\xi(U))$. So, given a section $\sigma_F \in \underline{\underline{\Omega}}^i(M)(F)$ we can extend it to a section $\sigma_U \in \underline{\underline{\Omega}}^i(M)(U)$ for U and open set with $F \subset U \subset M$. Now, we construct a function $\lambda : M \to [0,1]$ with $\lambda|_F = 1$ and support$(\lambda) \subset U$. To do this we use the fact that M is paracompact to show there is a collection of subsets $\{V_i \subset \bar{V}_i \subset U_i\}$ where both the U_i and V_i cover U, and thus F, and the U_i are locally finite. For each $V_i \subset \bar{V}_i \subset U_i$ we can construct a smooth bump function $\lambda_i : M \to [0,1]$ such that λ_i restricted to $\bar{V}_i$ is 1, and the support of λ_i is contained in U_i. Let $\hat{\lambda} = \sum_i \lambda_i$. Then the suppport of $\hat{\lambda}$ is contained in $\cup_i U_i \subset U$, and $\hat{\lambda}(x) \geq 1$ for $x \in F$. To get λ, compose $\hat{\lambda}$ with $\phi : [0,\infty) \to [0,1]$ a smooth function that is identically zero in some neighborhood of zero and identically 1 on $(1-\epsilon,\infty)$ for some $\epsilon > 0$. Then $\lambda = \phi \circ \hat{\lambda} : M \to [0,1]$ is a function with the desired properties. Now, we can extend our section σ_U to a global section by $\sigma' = \lambda\sigma_U \in \Omega^i(M)$, and we have $\sigma'|_F = \sigma_F$. Thus, we have shown,

Proposition 6.1.7. *The sheaves* $\underline{\underline{\Omega}}^i(M)$ *are soft.*

Now, we have the following basic lemma,

Lemma 6.1.8. *Suppose X is paracompact and we have the following short exact sequence of sheaves over X.*

$$0 \to \xi' \to \xi \to \xi'' \to 0$$

where ξ' is a soft sheaf. Then $H^0(\xi) \to H^0(\xi'') \to 0$ is exact.

Proof. Let $\sigma'' \in H^0(\xi'')$. Cover X by $\{V_i \subset \bar{V}_i \subset U_i\}_{i\in I}$ with $\{U_i\}_{i\in I}$ a locally finite open cover, and $\{V_i\}_{i\in I}$ an open covering. Let $\sigma_i \in \xi(U_i)$ be a lift of $\sigma''|U_i$. Now consider the collection of pairs $(J \subset I, \sigma_J \in \xi(\bigcup_{j\in J} \bar{V}_j))$, where $\sigma_J \mapsto \sigma''| \cup_{j\in J} \bar{V}_j$. Such pairs are partially ordered by inclusion of subsets and extension of sections. Any totally ordered chain in this partially ordered set has an upperbound: take the union of subsets, and use local finiteness to construct a section, so by Zorn's lemma there exists a maximal element, $(M \subset I, \sigma_M \in \xi(\bigcup_{j\in M} \bar{V}_j))$. It is left as an exercise to show that $M = I$. □

Corollary 6.1.9. *If ξ' and ξ are both soft, then so is ξ''*

Corollary 6.1.10. *If $0 \to \xi_0 \to \xi_1 \to \xi_2 \to \cdots$ is an exact sequence of soft sheaves then $0 \to H^0(\xi_0) \to H^0(\xi_1) \to H^(\xi_2) \to \cdots$ is an exact sequence.*

Corollary 6.1.11. *If ξ is a soft sheaf, then $H^i(\xi) = 0$ for $i > 0$.*

Thus, the double complex spectral sequence implies that $H^*(M;\underline{\underline{\mathbb{R}}}) = H^*_{dR}(M)$, completing the proof of theorem 6.1.4.

6.1.2 Čech cohomology and sheaf cohomology

Recall the definition of Čech cohomology with values in a sheaf. For a space X and an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ we defined the Čech cochains as,

$$\check{C}^k(\mathcal{U};\xi) = \prod_{\alpha_0\cdots\alpha_k} \xi(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}),$$

where we have choosen some total ordering of the index set A. The coboundary map $\delta : \check{C}^k \to \check{C}^{k+1}$ has the usual formula, and $\check{H}^*(\mathcal{U};\xi)$ is the cohomology of $(\check{C}^*(\mathcal{U};\xi),\delta)$. To define the Čech cohomology of the space we take the direct limit over all open covers,

$\varinjlim_{\text{open covers } \mathcal{U}} \check{H}^*(\mathcal{U};\xi) = \check{H}^*(X;\xi)$. The goal of this section is to identify $\check{H}^*(X;\xi)$ with the resolution cohomology, in the case where X is a paracompact space.

To do this, first we make the Čech cochains into a sheaf. We define a presheaf $\check{\mathcal{C}}^p(\mathcal{U};\xi)$, the Čech p-cochain presheaf on X by

$$\check{\mathcal{C}}^p(\mathcal{U};\xi)(U) = \check{C}^p(\mathcal{U}|_U;\xi|_U) = \prod_{\alpha_0\cdots\alpha_k} \xi(U \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

This clearly defines a presheaf, and in fact:

Lemma 6.1.12. *$\check{\mathcal{C}}^p(\mathcal{U};\xi)$ is a sheaf.*

Proof. Suppose that $U = \cup_i U_i$. We need to show that the following sequence is exact,

$$0 \longrightarrow \check{\mathcal{C}}^p(\mathcal{U};\xi)(U) \longrightarrow \prod_i \check{\mathcal{C}}^p(\mathcal{U};\xi)(U_i) \longrightarrow \prod_{i<j} \check{\mathcal{C}}^p(\mathcal{U};\xi)(U_i \cap U_j)$$

By definition, this is the same as showing the following sequence is exact,

$$\begin{aligned} 0 \longrightarrow \prod_{\alpha_0\cdots\alpha_k} \xi(U \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}) \longrightarrow & \prod_i \prod_{\alpha_0\cdots\alpha_k} \xi(U_i \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}) \\ \longrightarrow & \prod_{i<j} \prod_{\alpha_0\cdots\alpha_k} \xi(U_i \cap U_j \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}) \end{aligned}$$

In this sequence we can interchange the order in which we take products. Then for a given $(p+1)$-tuple the sequence is exact by the sheaf property of ξ,

$$\begin{aligned} 0 \longrightarrow \xi(U \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}) \longrightarrow & \prod_i \xi(U_i \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}) \\ \longrightarrow & \prod_{i<j} \xi(U_i \cap U_j \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}) \end{aligned}$$

Since the product of exact sequences is exact, this proves the result. $\square$

Now, consider the canonical resolution,

$$0 \longrightarrow \xi \longrightarrow C^0(\xi) \longrightarrow C^1(\xi) \longrightarrow C^2(\xi) \longrightarrow \cdots$$

For each sheaf in this resolution we make the Čech cochains into a sheaf as described and we have,

$$\begin{tikzcd}
 & {} & {} & {} & & \\
 & \check{\mathcal{C}}^1(\mathcal{U};\xi) \arrow[u] & \check{\mathcal{C}}^1(\mathcal{U};C^0(\xi)) \arrow[u] & \check{\mathcal{C}}^1(\mathcal{U};C^1(\xi)) \arrow[u] & & \\
 & \check{\mathcal{C}}^0(\mathcal{U};\xi) \arrow[u] & \check{\mathcal{C}}^0(\mathcal{U};C^0(\xi)) \arrow[u] & \check{\mathcal{C}}^0(\mathcal{U};C^1(\xi)) \arrow[u] & & \\
0 \arrow[r] & \xi \arrow[u] \arrow[r] & C^0(\xi) \arrow[u] \arrow[r] & C^1(\xi) \arrow[u] \arrow[r] & C^2(\xi) \arrow[r] & \cdots
\end{tikzcd} \tag{3}$$

We want to show three things,

1. There are horizontal arrows in this diagram.
2. The vertical complexes are resolutions.
3. The $\check{\mathcal{C}}^*(C^i(\xi))$ have no higher cohomology.

Then puting these together, we will show that the cohomology of the total complex equals the cohomology obtained from the canonical resolution, and thus we will have identified the sheaf cohomology with the Čech cohomology.

To show that we have horizontal maps, suppose that we have map $f : \xi_1 \to \xi_2$. then there is an induced map

$$\check{\mathcal{C}}^p(f) : \check{\mathcal{C}}^p(\mathcal{U}; \xi_1) \to \check{\mathcal{C}}^p(\mathcal{U}; \xi_2).$$

We need a map $\check{\mathcal{C}}^p(\mathcal{U}; \xi_1)(U) \to \check{\mathcal{C}}^p(\mathcal{U}; \xi_2)(U)$.

$$\textstyle\prod_{\alpha_0 \cdots \alpha_p} \xi_1(U \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}) \xrightarrow{\ f\ } \prod_{\alpha_0 \cdots \alpha_p} \xi_2(U \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$$

These maps of sheaves commute with δ.

For step two, we have the following lemma,

Lemma 6.1.13. *For any sheaf ξ we have a resolution*

$$0 \to \xi \to \check{\mathcal{C}}^0(\mathcal{U}; \xi) \to \check{\mathcal{C}}^1(\mathcal{U}; \xi) \to \check{\mathcal{C}}^2(\mathcal{U}; \xi) \to \cdots$$

Proof. This is a resolution, since the sequence is exact on the stalk level. □

For step three, we have,

Lemma 6.1.14. *If ξ is flabby, then $\check{\mathcal{C}}^p(\mathcal{U}; \xi)$ is also flabby (for all p and for all $\mathcal{U}$).*

Proof. Let $\sigma \in \check{\mathcal{C}}^p(\mathcal{U}; \xi)$. Then

$$\sigma = \prod_{\alpha_0 \cdots \alpha_p} \sigma_{\alpha_0, \ldots, \alpha_p},$$

where $\sigma_{\alpha_0, \ldots, \alpha_p} \in \xi(U \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$. Since ξ is flabby we can extend each one of these $\sigma_{\alpha_0, \ldots, \alpha_p}$ to $\hat{\sigma}_{\alpha_0, \ldots, \alpha_p} \in \xi(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$, and thus extend σ to

$$\hat{\sigma} = \prod_{\alpha_0 \cdots \alpha_p} \hat{\sigma}_{\alpha_0, \ldots, \alpha_p}.$$

□

Now, we have that each vertical column in the diagram (3) is a flabby resolution,

$$\begin{tikzcd}
{} \\
\check{\mathcal{C}}^1(C^i(\xi)) \arrow[u] \\
\check{\mathcal{C}}^0(C^i(\xi)) \arrow[u] \\
C^i(\xi) \arrow[u]
\end{tikzcd}$$

and thus,

$$H^*(\check{\mathcal{C}}^*(C^i(\xi)),\delta) = \begin{cases} 0 & *>0 \\ H^0(C^i(\xi)) & *=0 \end{cases}$$

Now, we use the double complex $\oplus_{p,q} A^{p,q}$ with

$$A^{p,q} = H^0(\check{\mathcal{C}}^q(\mathcal{U}; C^p(\xi))).$$

and maps given by,

$$\begin{tikzcd}
A^{p,q+1} & \\
A^{p,q} \arrow[u, "\delta-\text{Čech}"'] \arrow[r, "d"] & A^{p+1,q}
\end{tikzcd}$$

So, our double complex looks like,

$$\begin{tikzcd}
 & {} & {} & {} & \\
 & \check{\mathcal{C}}^1(\mathcal{U};\xi) \arrow[u, "\delta"'] \arrow[r] & \check{\mathcal{C}}^1(\mathcal{U};C^0(\xi)) \arrow[u, "\delta"'] \arrow[r] & \check{\mathcal{C}}^1(\mathcal{U};C^1(\xi)) \arrow[u, "\delta"'] \arrow[r] & {} \\
 & \check{\mathcal{C}}^0(\mathcal{U};\xi) \arrow[u, "\delta"'] \arrow[r] & \check{\mathcal{C}}^0(\mathcal{U};C^0(\xi)) \arrow[u, "\delta"'] \arrow[r] & \check{\mathcal{C}}^0(\mathcal{U};C^1(\xi)) \arrow[u, "\delta"'] \arrow[r] & {} \\
0 \arrow[r] & \xi \arrow[u] \arrow[r] & C^0(\xi) \arrow[u] \arrow[r] & C^1(\xi) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Now, filtering on p,

$$E_0^{p,q} = H^0(\check{\mathcal{C}}^q(\mathcal{U}; C^p(\xi)))$$

and $d_0 = \delta$.

$$E_1^{p,q} = \begin{cases} 0 & q>0 \\ H^0(C^p(\xi) & q=0 \end{cases}$$

and $d_1 : H^0(C^p(\xi)) \to H^0(C^{p+1}(\xi))$.

$$E_2^{p,q} = \begin{cases} 0 & q > 0 \\ H^p_{\text{resolution}}(\xi) & q = 0 \end{cases}$$

and thus,

$$E_2^{p,q} = E_\infty^{p,q} = H^p(C^*_{\text{total}}, D) = H^p_{\text{resolution}}(\xi).$$

Now, we filter on q. Then $E_0^{p,*}$ are the global sections,

$$\check{C}^p(\mathcal{U}; C^0(\xi)) \xrightarrow{\quad d \quad} \check{C}^p(\mathcal{U}; C^1(\xi)) \xrightarrow{\quad d \quad} \check{C}^p(\mathcal{U}; C^2(\xi))$$

and $d_0 : E_0^{p,*} \to E_0^{p+1,*}$. Now,

$$\check{C}^p(\mathcal{U}; C^i(\xi)) = \prod_{(a_0,\dots,a_p)} C^i(\xi)(U_{a_0} \cap \dots \cap U_{a_p}).$$

and the map $d : \check{C}^p(\mathcal{U}; C^i(\xi)) \to \check{C}^p(\mathcal{U}; C^{i+1}(\xi))$ is given by,

$$\prod_{(a_0,\dots,a_p)} \Big(d : C^i(\xi)(U_{a_0} \cap \dots \cap U_{a_p})) \to C^{i+1}(\xi)(U_{a_0} \cap \dots \cap U_{a_p})) \Big).$$

So,

$$E_1^{p,q} = \prod_{(a_0,\dots,a_p)} \mathcal{H}^q(C^*(\xi)(U_{a_0} \cap \dots \cap U_{a_p})).$$

where $\mathcal{H}^q$ is a cohomology presheaf of $C^*(\xi)$,

$$\mathcal{H}^q(C^*(\xi))(U) = \frac{\operatorname{Ker}(C^q(\xi)(U) \to C^{q+1}(\xi)(U))}{\operatorname{Im}(C^{q-1}(\xi)(U) \to C^q(\xi)(U))}.$$

Notice that $\mathcal{H}^q(C^*(\xi))$ is a presheaf whose associated sheaf is trivial, and $\mathcal{H}^0(C^*(\xi)) = \xi$. A presheaf $\mathcal{A}$ determines the trivial sheaf if and only if for every $x \in X$, $\varinjlim_{x \in U} \mathcal{A}(U) = \{0\}$.

In our case, for $q > 0$,

$$\begin{aligned}
\varinjlim_{\{U \mid x \in U\}} \mathcal{H}^q(C^*(\xi))(U) &= \varinjlim_{\{U \mid x \in U\}} \frac{\operatorname{Ker}(C^q(\xi)(U) \to C^{q+1}(\xi)(U))}{\operatorname{Im}(C^{q-1}(\xi)(U) \to C^q(\xi)(U))} \\
&= \frac{\operatorname{Ker}\Big(\varinjlim_{\{U \mid x \in U\}} C^q(\xi)(U) \to \varinjlim_{\{U \mid x \in U\}} C^{q+1}(\xi)(U) \Big)}{\operatorname{Im}\Big(\varinjlim_{\{U \mid x \in U\}} C^{q-1}(\xi)(U) \to \varinjlim_{\{U \mid x \in U\}} C^q(\xi)(U) \Big)} \\
&= \{0\}
\end{aligned}$$

Now, suppose we have an open cover $\mathcal{V} = \{V_b\}_{b \in B}$ that refines $\mathcal{U} = \{U_a\}_{a \in A}$. and a refinement function $r : B \to A$. Then r induces a map on the double complex of sheaves and hence on the double complex of global sections.

$$\check{C}^p(\mathcal{U}; C^q(\xi)) \to \check{C}^p(\mathcal{V}; C^q(\xi))$$

$$\sigma = \prod \sigma_{a_0,\dots,a_p} \mapsto \tau = \prod \tau_{b_0,\dots,b_p}$$

where,

$$\tau_{b_0,\dots,b_p} = \sigma_{r(b_0),\dots,r(b_p)}|_{V_{b_0} \cap \cdots \cap V_{b_p}}.$$

The map on the double complex depends on the refinelment. filtering on the resolution degree,

$$E_1^{p,q} = \begin{cases} 0 & q > 0 \\ H^0(C^p(\xi)) & q = 0 \end{cases}$$

For any choice of r the induced map on $E_1^{p,q}$ is the identity, and all possible refinements induce the same isomorphism on the cohomology of the double complex.

Now, we filter on the Čech degree. Here things are much more complicated, and more interesting. We have,

$$E_1^{p,q}(\mathcal{U}) = \check{C}^p(\mathcal{U}; \mathcal{H}^q(C^*(\xi)))$$
$$E_2^{p,q}(\mathcal{U}) = H^p(\mathcal{U}; \mathcal{H}^q(C^*(\xi)))$$

and the map $r : \mathcal{V} \to \mathcal{U}$ induces a map

$$E_1^{p,q}(\mathcal{U}) \xrightarrow{r^*} E_1^{p,q}(\mathcal{V})$$

$$\sigma = \prod \sigma_{a_0,\dots,a_p} \mapsto \tau = \prod \tau_{b_0,\dots,b_p}$$

where,

$$\tau_{b_0,\dots,b_p} = \sigma_{r(b_0),\dots,r(b_p)}|_{V_{b_0} \cap \cdots \cap V_{b_p}}.$$

All the refinement maps induce the same map on $E_2^{p,q}$. In this spectral sequence,

$$E_2^{p,0} = H^p(\mathcal{U}; \mathcal{H}^0(C^*(\xi))) = H^p(\mathcal{U}; \xi) = \varinjlim_{\text{open covers}} (E_2^{p,0}) = \check{H}^p(X; \xi)$$

Theorem 6.1.15. $\varinjlim_{\textit{open covers}} (E_2^{p,0}) = 0$ *for* $q > 0$ *if* X *is paracompact.*

Proposition 6.1.16. *Suppose that* $\mathcal{A}$ *is a presheaf whose associated sheaf is trivial (X paracompact), then* $\check{H}^*(X; \mathcal{A}) = 0$ *for all* $*$.

Now, $\varinjlim_{\text{open covers}} (E_2^{p,q}) = 0$ for $q > 0$, so the direct limit spectral sequence has,

$$E_2^{p,q} = 0 \text{ for } q > 0$$

and,

$$E_2^{p,0} = \check{H}^p(X;\xi) = E_\infty^{p,0} = H^p(\text{total complex})$$

and converges to $H^*(\text{total complex})$.

Corollary 6.1.17. DeRham's Theorem *For a smooth manifold M, the deRham cohomology $H^*_{dR}(M)$ is identified with the Čech cohomology with real coefficents (i.e. coefficents in the constant sheaf $\underline{\underline{\mathbb{R}}}$)*

7 Applications to Manifolds

Manifolds are one of the most important classes of topological spaces in mathematics, and the tools of algebraic topology have been used extensively to study their properties. A great deal is known about the algebraic topological invariants that we have developed so far in the case where the space under consideration is a manifold. The starting point for this application is the Poincaré Duality theorem, which relates the homology and cohomology groups associated to a compact manifold. For our purposes we will generally stick to the case of smooth manifolds, since this will allow us to make use of Morse theory; however, many of these results also hold in non-smooth categories.

7.1 Morse Theory Basics

Morse theory studies the topology of a smooth n-manifold M by looking at smooth functions from the manifold to $\mathbb{R}$. We will apply the results of Morse theory to prove Poincaré Duality. There is an excellent book on Morse theory by John Milnor, which contains most of the results that we will need, so here we will only state the main results and refer the reader to Milnor's book for details.

Definition 7.1.1. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if $df \in \Gamma(T^*M)$ has isolated transverse zeros.

Let $f : M \to \mathbb{R}$ be a Morse function. In local coordinates $(x^1, \ldots, x^n)$, centered at a point $x \in M$, $df(x) = \sum \frac{\partial f}{\partial x^i}(0)dx^i = 0$ if and only if $\frac{\partial f}{\partial x^i}(0) = 0$ for all i. Such points are called *critical points*. Points which are not critical points are called *regular points*.

Since f is a Morse function, $D(df)_{(x,0)}$ is transverse to $TM_{(x,0)}$. When this condition is met, the critical point is said to be *non-singular* or *non-degenerate*. The condition that a critical point be non-degenerate is the same as that the Hessian of f, $H(f)$, be non-singular. The Hessian of f is the symmetric matrix given by,

$$H(f)_{ij}(0) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(0) \right)$$

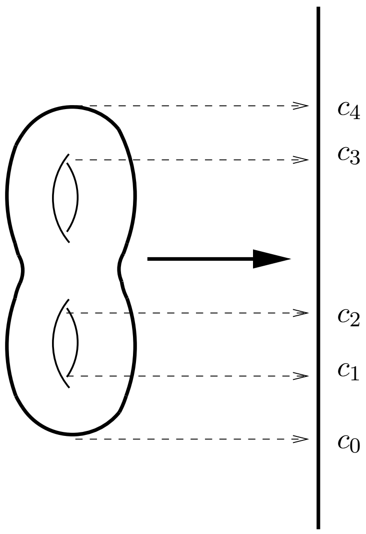

Figure 14: A Morse function on Σ_2 and its critical points

The Hessian gives a symmetric bilinear, or quadratic form.

Lemma 7.1.2. *Every real quadratic form on $\mathbb{R}^n$ is equivalent to a diagonal form with $+1, -1, 0$ on the diagonal. The form is non-degenerate iff there are no zeroes on the diagonal. The number of $+1$'s and -1's are invariants of the form.*

The number of -1's on the diagonal of a quadratic form is called the *index* of the form. If x is a non-degenerate critical point of f then the *index* of x is the index of the quadratic form given by $D(df)_x = \Big(H(f)(x)\Big)$.

Theorem 7.1.3. *Suppose that p is a non-degenrate critical point for f, then there exist local coordintes $(x^1, \dots, x^n)$ centered at p such that $f(x^1, \dots, x^n) = f(p) + \sum_{i=0}^{k}(x^i)^2 - \sum_{j=k+1}^{n}(x^j)^2$. If p is a regular point then there exist local cooordintes $(x^1, \dots, x^n)$ centered at p such that $f(x^1, \dots, x^n) = f(p) + x^1$.*

Now we completely understand our Morse function locally. Near a critical point p there exist local coordinates $\{\vec{x}, \vec{y}\}$ such that $f(\vec{x}, \vec{y}) = f(p) - |\vec{x}|^2 + |\vec{y}|^2$. Near a regular point q there exist local coordinates $\{x^1, \dots, x^n\}$ such that $f(\vec{x}) = f(q) + x^1$. The regular points of a Morse function $f : M \to \mathbb{R}$ are an open dense subset of M. The critical points are a discrete subset of M. If M is compact this implies that there are only finitely many critical points. Our next goal is to understand $f^{-1}([a, b])$ for a and b sufficently close.

Definition 7.1.4. A *gradient-like* vector field for f, is a vector field V on M such that

1. $< df(x), V(x) > > 0$ for every regular point x.

2. If p is a critical point, then there exist local coordinates $(\vec{x}, \vec{y})$ such that $f(\vec{x}, \vec{y}) = f_0 + |\vec{y}|^2 - |\vec{x}|^2$ and $V = -2\sum x^i \frac{\partial}{\partial x^i} + 2\sum \frac{\partial}{\partial y^i}$.

Lemma 7.1.5. *f has gradient like vector fields.*

Theorem 7.1.6. *Let $f : M \to \mathbb{R}$ be a Morse function, $a < b \in \mathbb{R}$ and suppose that there is no crtical point $x \in M$ with $f(x) \in [a, b]$. Then $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$*

Remark 7.1.7. If a is a regular value, then $f^{-1}(a)$ is a smooth codim 1 submanifold of M, and $f|_{f^{-1}([a,b])}$ = projection to the second factor, i.e. $f^{-1}(t) \cong f^{-1}(a)$ for all $a \leq t \leq b$.

Now assume there exists a single critical point c such that $a \leq f(c) \leq b$ and furthermore, $a < f(c) < b$. Our goal is to model the topology of $f^{-1}([a, b])$, in terms of $f^{-1}(a)$ and information about c. Let V be a gradient like vector field for f. For a point $x \in M$ let $\gamma_x : \mathbb{R} \to M$ be a flow line for V starting at x.

Definition 7.1.8. The *descending* or *stable* manifold of a critical point $c \in M$ is $\{x \in M | \lim_{t\to\infty} \gamma_x(t) = c\}$. The *ascending* or *unstable* manifold of a critical point $c \in M$ is $\{x \in M | \lim_{t\to-\infty} \gamma_x(t) = c\}$.

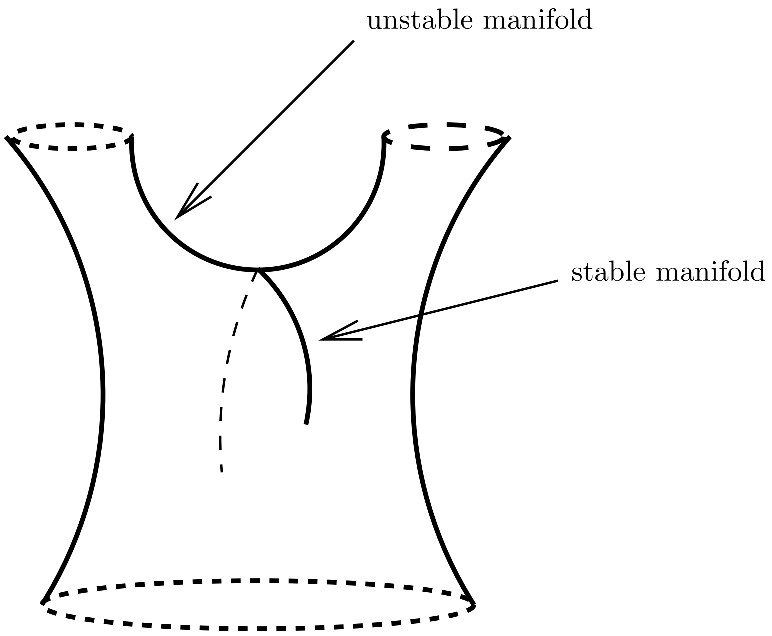

Figure 15: The stable and unstable manifolds

If c is a critical point of index i then the stable manifold is diffeomorphic to $\mathbb{R}^i$.

Theorem 7.1.9. *If there exists a unique critical point with value in $[a, b]$, that being a critical point of index k, then*

$$H_*(f^{-1}([a,b]), f^{-1}(a)) = \begin{cases} \mathbb{Z} & * = k \\ 0 & * \neq k \end{cases}$$

If there exists a unique critical point with value in $[a, b]$, that being a critical point of index k, then

$$H_*(M_{(-\infty,b]}, M_{(-\infty,a]}) = H_*(f^{-1}_{[a,b]}, M_b) = \begin{cases} \mathbb{Z} & * = k \\ 0 & * \neq k \end{cases}$$

Thus, $H_*(M_{(-\infty,b]}) = H_*(M_{(-\infty,a]})$ for $* \neq k-1, k$, and for these cases we have the following exact sequence,

$$0 \to H_k(M_{(-\infty,b]}) \to H_k(M_{(-\infty,a]}) \to \mathbb{Z} \to H_{k-1}(M_{(-\infty,b]}) \to H_{k-1}(M_{(-\infty,a]}) \to 0$$

Corollary 7.1.10. *If M is a compact smooth manifold then $H_*(M)$ is finitely generated.*

Theorem 7.1.11. Morse Inequalities *There is a free abelian chain complex whose chain group in degree k has as a basis the set of critical points of index k, and whose homlogy is isomorphic to the singular homology of M.*

Corollary 7.1.12. *If M^n has a Morse function with exactly two critical points, then $H_*(M) \cong H_*(S^n)$.*

7.2 The Fundamental Class

Suppose that M is a closed oriented n-manifold. Recall that

$$H_*(M, M \setminus \{x\}) = \begin{cases} \mathbb{Z} & * = n \\ 0 & * \neq n \end{cases}$$

Furthermore, an orientation of M at x determines a generator for $H_n(M, M \setminus \{x\})$. A global orientation for M determines a generator $o_x \in H_n(M, M \setminus \{x\})$ for every $x \in M$. An *orientation homology class* or *fundamental class* is an element $[M] \in H_n(M)$ such that $i_*([M]) = o_x \in H_n(M, M \setminus x)$ for every $x \in M$, where $i : (M, \emptyset) \hookrightarrow (M, M \setminus \{x\})$ is the inclusion. For a manifold with boundary,

$$H_n(M, M \setminus \{x\}) = \begin{cases} \mathbb{Z} & * = n \\ 0 & * \neq n \end{cases}$$

if $x \in \text{int}(M)$. If $x \in \partial M$, then $M \setminus \{x\} \hookrightarrow M$ induces an isomorphism on homology. A homology orientation class for M is $[M, \partial M] \in H_n(M, \partial M)$ whose image in $H_n(M, M \setminus \{x\})$ is o_x for all $x \in \text{int}(M)$.

Theorem 7.2.1. *If M is oriented then it has a homology orientation giving that orientation.*

Lemma 7.2.2.

$$H_*(S^k \times D^{n-k}, S^k \times \partial D^{n-k}) = \begin{cases} \mathbb{Z} & * = n, n-k \\ 0 & \text{otherwise} \end{cases}$$

The relative class in degree $n-k$ is carried by $(\{x\} \times D^{n-k}, \{x\} \times \partial D^{n-k})$ and a generator of H_n is a homology orientation. Also,

$$H_*(S^k \times S^{n-k-1}) = \begin{cases} \mathbb{Z} & * = 0, k, n-k-1, n-1 \\ 0 & \text{otherwise} \end{cases}$$

The classes in degree k and $n-k-1$ are carried by $S^k \times \{y\}$ and $\{x\} \times S^{n-k-1}$. A generator for H_{n-1} is a homology orientation.

Lemma 7.2.3. *Let M be a smooth n-manifold without boundary and $X \subset M$ an n-submanifold, possibly with boundary, such that $\partial X \subset M$ is a codim 1 submanifold. Then there is a diffeomorphism $(-\epsilon, \epsilon) \times \partial X \hookrightarrow M$ whose image is a neighborhood of ∂X and such that the intersection with X is $(-\epsilon, 0] \times \partial X$.*

Proof. There exists a neighborhood U of ∂X and a vector field V on U such that for every $x \in \partial X$ V_x points out of X. Use local coordinates to define V near each x, then add these together with a partition of unity. Integrate the vector field to obtain,

$$(-\epsilon, \epsilon) \times \partial X \xrightarrow{\Phi} M$$

given by $(t, x) \mapsto \gamma_x(t)$ where $\gamma_x(t)$ is a flow line for V with $\gamma_x(0) = x$. Then $D\Phi_{(0,x)}$ is a linear isomorphism, so $\Phi : (\epsilon, \epsilon) \times \partial X \to M$ is a local diffeomorphism. If we can show that it is one-to-one then it will be a diffeomorphism onto its image. If $\Phi|_{(-\delta,\delta)\times\partial X}$ is never one-to-one then there exist sequences $(x_n, t_n) \neq (y_n, s_n)$ with $t_n \to 0$, $s_n \to 0$ and $x_n, y_n \in \partial X$ with $\Phi(x_n, t_n) = \Phi(y_n, s_n)$. Pass to a subsequence so that $x_n \to x$ and $y_n \to y$ then $\Phi(x, 0) = \Phi(y, 0)$, which implies $x = y$, and thus both $\Phi(x_n, t_n) \to \Phi(x, 0)$ and $\Phi(y_n, s_n) \to \Phi(x, 0)$, which is a contradiction since Φ is a local diffeomorphism. □

Corollary 7.2.4. *There is a homotopy $H : C_1 \times I \to C_1$ such that $H_0 =$Id, $H_t|\partial C_1 =$Id, and $H_1(C_{1/2}) = \partial X$, where C_1 is a collar neighborhood of ∂X and $C_{1/2}$ is a smaller collar neighborhood.*

We now establish a slightly modified form of Mayer-Vietoris.

Proposition 7.2.5. *Let $X = A \cup_C B$ where $A, B \subset X$ are closed subsets and $A \cap B = C$. Let $U_A \supset A$ and $U_B \supset B$ be open sets in X. Suppose there is a homotpy $H : X \times I \to X$ with $H_0 = \mathrm{Id}_X$ and $H_t(A) \subset A$, $H_t(B) \subset B$ for all t and $H_1(U_A) \subset A$, $H_1(U_b) \subset B$. Then any homology class in X has a cycle representative $\zeta_A + \zeta_B$ where ζ_A is a chain in A and ζ_B is a chain in B and $\partial\zeta_A = -\partial\zeta_B \in S_*(C)$.*

Example 7.2.6. Let M be a closed n-manifold and $X \subset M$ a codim 0 submanifold, possibly with boundary. Let $A = X$, $B = M \setminus \mathrm{int}(X)$ and $C = \partial X$. Then the corollary above provides the neccesary homotopy and the proposition implies that any homology class in X has a cycle representative $\zeta_A + \zeta_B$ where ζ_A is a chain in A and ζ_B is a chain in B and $\partial\zeta_A = -\partial\zeta_B \in S_*(C)$.

Proof. Given $\alpha \in H_k(X)$ take a representative which is small with respect to the open cover $\{U_A, U_B\}$. Say $\zeta_{U(A)} + \zeta_{U(B)}$ is such a cycle representative for α with $\zeta_{U(A)} \in S_*(U_A)$ and $\zeta_{U(B)} \in S_*(U_B)$. Consider $(H_1)_*(\zeta_{U(A)} + \zeta_{U(B)})$. This is a new cycle representative for the same class. Letting $\zeta_A = (H_1)_*(\zeta_{U(A)}) \in S_*(A)$ and $\zeta_B = (H_1)_*(\zeta_{U(B)}) \in S_*(B)$, we have a representative for α as desired. □

Theorem 7.2.7. *If M is a closed compact orientable n-manifold then M has a unique orientation class $[M] \in H_n(M)$.*

Proof. Let $f : M \to \mathbb{R}$ be a Morse function with critical values $c_1 < \cdots < c_l$ such that $f^{-1}(c_i)$ has a unique critical point for each i. Take $a_0 < c_1 < a_1 < c_2 < \cdots < c_l < a_l$. We prove by induction that $M_{(-\infty,a_i]} = f^{-1}(-\infty, a_i]$ has a unique orientatation class. First, $f^{-1}(-\infty, a_0] = \emptyset$, so there is nothing to show. For $f^{-1}(-\infty, a_0] = D^n$ we have $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ and $H_n(D^n, S^{n-1}) \to H_n(D^n, D^n \setminus \{x\})$ is an isomorphism, $[D^n, S^{n-1}] \mapsto o_x$.

Now, let $M_- = M_{(-\infty,a_{i-1}]}$ and $M_+ = M_{(-\infty,a_i]}$. Suppose that there exists a unique orientation class $[M_-, \partial M_-] \in H_n(M_-, f^{-1}(a_{i-1}) = \partial M_-)$. Let $M' = M_- \cup_A A \times I$ where $A = M_{a_i-1} \setminus \mathrm{int}(S^{k-1} \times D^{n-k})$. Take a chain representative ζ_- for $[M_-, \partial M_-]$. Then $\partial\zeta_-$ is homologus to $\zeta_A + \zeta_B$ where $\zeta_A \in S_{n-1}(A)$ and $\zeta_B \in S_{n-1}(S^{k-1} \times D^{n-k})$. So $\partial\zeta_- = \zeta_A + \zeta_B + \partial\mu$ for some $\mu \in S_n(\partial M_-)$. Replace ζ_- by $\zeta_- - \mu \in S_n(M_-)$. This still represents $[M_-, \partial M_-]$, so $\partial\zeta_- = \zeta_A + \zeta_B$. By subdividng we have $\zeta_- + \zeta_A \times I \in S_n(M')$, and

$$\partial(\zeta_- + \mathrm{sub}(\zeta_A \times I)) = \zeta_A + \zeta_B + \zeta_A \times \{1\} - \zeta_A \times \{0\} + \mathrm{sub}(\partial\zeta_A \times I) = \zeta_B + \zeta_A \times \{1\} - \mathrm{sub}(\partial\zeta_A \times I).$$

We have $M_+ = M_- \cup D^n$, and fundamental classes $[M_-, \partial M_-]$, $[D^n, \partial D^n]$ inducing opposite orientations on the boundary. We have $D^n \cap M = B \cup_{\partial B = \partial A} (\partial A \times I)$. Choose chain representatives ζ' for $[M', \partial M']$ and ζ_D for $[D^n, \partial D^n]$ such that $\partial\zeta' = \mu_o + \mu_1$ and $\partial\zeta_D = \nu_0 + \nu_1$ with $\mu_0, \nu_0 \in S_{n-1}(B \cup \partial A \times I)$. Then μ_o and ν_0 give appropriate orientations, and hence $[\mu_0] = -[\nu_0] \in H_{n-1}(B \cup \partial A \times I, \partial)$ so $\mu_0 + \nu_0 = \partial\gamma$ for some $\gamma \in S_n(B \cup \partial A \times I)$. By adding γ we can make $\mu_0 = -\nu_0$. Now, let $\zeta_+ = \zeta_D + \zeta' \in S_n(M_+)$. Then $\partial(\zeta_D + \zeta') = \mu_1 + \nu_1 \in S_{n-1}(\partial M_+)$. Then ζ_+ is a relative cycle for $(M_+, \partial M_+)$ and agrees in M_- with ζ_-, so ζ_+ induces the orientation $o_x \in H_n(M_+, M_+ \setminus \{x\})$ for all $x \in \mathrm{int}(M_-)$ and hence for all $x \in \mathrm{int}(M_+)$. □

7.3 Poincaré Duality

Poincaré duality asserts that certain homology and cohomology classes of a manifold are isomorphic. The isomorphism is given by taking cap product with the fundamental class of the manifold.

Theorem 7.3.1. Poincaré Duality *Let M be a closed oriented n-manifold. Let $[M] \in H_n(M)$ be the orientation class, then $\cap[M] : H^i(M;\mathbb{Z}) \to H_{n-i}(M;\mathbb{Z})$ is an isomorphism.*

Theorem 7.3.2. Lefschetz Duality *If M is a manifold with boundary then $\cap[M,\partial M] : H^i(M;\mathbb{Z}) \to H_{n-i}(M,\partial M;\mathbb{Z})$ is an isomorphism.*

The power of this theorem is illustrated by the following immediate corollaries.

Corollary 7.3.3. *$H_k(m) = 0$ for all $k > n$.*

Corollary 7.3.4. *$H_n(M) \cong \mathbb{Z}$ and $[M]$ is a generator.*

Corollary 7.3.5. $\mathrm{rk}(H_i(M)) = \mathrm{rk}(H_{n-i}(M))$.

Corollary 7.3.6. $\mathrm{Tor}H_i(M) \cong \mathrm{Tor}H_{n-i-1}(M)$.

Corollary 7.3.7. *$H_{n-1}(M)$ is torsion free.*

Remark 7.3.8. This is not true if M is not oriented. For example $H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$.

Proof. Poincaré Duality Take a Morse function $f : M \to \mathbb{R}$ such that if c is a critical value, $f^{-1}(c)$ has exactly one critical point. Let $c_0 < c_1 < c_2 < \cdots < c_N$ be the critical values of f and let $x_i \in f^{-1}(c_i)$ be the citical points. Choose $a_0 < c_0, a_1 < c_1 < \cdots < a_N < c_N < a_{N+1}$. Let $M_j = f^{-1}((-\infty, a_j])$. Then each M_j is a compact manifold with boundary. Since M is oriented so are $\mathrm{int}(M_j)$ and $\partial M_j = f^{-1}(\{a_j\})$. $M_0 = \emptyset$, $M_{N+1} = M$ and M_{j+1} is obtained from M_j by adding a single handle.

Lemma 7.3.9. *There is a relative fundamental class $[M_j, \partial M_j] \in H_N(M_j, \partial M_j)$, and $\cap[M_j, \partial M_j] : H^i(M_j) \to H_{n-i}(M_j, \partial M_j)$ is an isomorphism for all i.*

Proof. The proof is by induction on j. The statement for $j = N+1$ is Poincaré Duality for M. Let $X_{j+1} = M_{j+1} \setminus \mathrm{int}(M_j) = f^{-1}([a_j, a_{j+1}])$. We have already shown the existence of a fundamental class $[M_j, \partial M_j] \in H_n(M_j, \partial M_j)$ such that under the composition,

$$H_n(M_{j+1}, \partial M_{j+1}) \xrightarrow{i_*} H_n(M_j, X_{j+1}) \xrightarrow[excision]{\cong} H_n(M_j, \partial M_j)$$

$[M_{j+1}, \partial M_{j+1}]$ maps to $[M_j, \partial M_j]$. We have the exact sequence,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_i(X_{j+1}, \partial M_{j+1}) & \xrightarrow{i_*} & H_i(M_{j+1}, X_{j+1}) & \longrightarrow & H_{i-1}(X_{j+1}, \partial M_{j+1}) & \longrightarrow & \cdots \\ & & & & \| & & & & \\ & & & & H_i(M_j, \partial M_j) & & & & \end{array}$$

We have seen that $H_*(X_{j+1}, \partial M_j) = H_*(D^k, \partial D^k)$ where k is the index of f at x_{j+1}, and X_{j+1} deformation retracts onto the stable manifold at x_j union ∂M_j which equals $D^k \cup \partial M_j$.

Now consider $g = -f$. Then $g : X_{j+1} \to [-a_{j+1}, -a_j]$. The lower boundary under g is $g^{-1}(-a_{j+1}) = f^{-1}(a_{j+1}) = \partial M_{j+1}$. There is still a critical point for g at x_{j+1}, but the index of that critical point is n minus the index of x_{j+1} for f. So, we see

$$H_*(X_{j+1}, \partial M_{j+1}) = H_*(\text{unstable}_f(x_{j+1}) \cup \partial M_{j+1}, \partial M_{j+1}) = \begin{cases} \mathbb{Z} & * = n-k \\ 0 & \text{otherwsie} \end{cases}$$

We conclude that $i_* : H_i(M_{j+1}, \partial M_{j+1}) \to H_I(M_j, \partial M_j)$ is an isomorphism for $i \neq n-k+1, n-k$. For these cases we have,

$$0 \to H_{n-k+1}(M_{j+1}, \partial M_{j+1}) \to H_{n-k+1}(M_j, \partial M_j) \to \mathbb{Z} \to H_{n-k}(M_{j+1}, \partial M_{j+1}) \to H_{n-k}(M_j, \partial M_j) \to 0$$

Now, compare $H^*(M_{j+1})$ and $H^*(M_j)$. We have,

$$\begin{tikzcd} \cdots \arrow[r] & H^*(M_{j+1}, \partial M_j) \arrow[r] & H^*(M_{j+1}) \arrow[r, "i^*"] & H^*(M_j) \arrow[r] & H^{*+1}(M_{J+1}, M_j) \arrow[r] \arrow[d, equal] & \cdots \\ & & & & H^{*+1}(X_{j+1}, \partial M_j) \arrow[d, equal] & \\ & & & & H^{*+1}(D^k, \partial D^k) & \end{tikzcd}$$

So, $i^* : H^l(M_{j+1} \to H^l(M_j)$ is an isomorphism for $l \neq k, k-1$. For these we have,

$$0 \to H^{k-1}(M_{j+1}) \to H^{k-1}(M_j) \to \mathbb{Z} \to H^k(M_{j+1}) \to H^k(M_j) \to 0$$

First, we look away from the critical dimensions, $i \neq k, k-1$.

$$\begin{tikzcd} H^i(M_{j+1}) \arrow[r, "\cong", "i_*"'] \arrow[d, "\cap[M_{j+1}, \partial M_{j+1}]"] & H^i(M_j) \arrow[d, "\cap[M_j, \partial M_j]", "\cong"'] \\ H_{n-i}(M_{j+1}, \partial M_{j+1}) \arrow[r, "\cong"] & H_{n-i}(M_j, \partial M_j) \arrow[d, equal] \\ & H_{n-i}(M_{j+1}, X_{j+1}) \end{tikzcd}$$

So, all we need to show is that the diagram commutes and we will have shown that Lefshchetz duality holds for M_{j+1} for all degrees except $i = k-1, k$. To prove that the diagram commutes, take a cycle representative for the fundamental class of M_{j+1} of the form $\zeta_j + \chi_{j+1}$ where $\zeta_j \in S_n(M_j)$ is a cycle representative for $[M_j, \partial M_j]$ and $\chi_{j+1} \in S_n(X_{j+1})$. Then $\partial \chi_{j+1} = \partial_+ \chi_{j+1} + \partial_- \chi_{j+1}$ where $\partial_+ \chi_{j+1} \in S_{n-1}(\partial M_{j+1})$, $\partial_- \chi_{j+1} \in S_{n-1}(\partial M_j)$

and $\partial\zeta_j = -\partial_-\chi_{j+1}$. Suppose $\zeta_j = \sum n_\sigma \sigma$ and $\chi_{j+1} = \sum n_\tau \tau$. Start with a cocylce $\mu \in S^i(M_{j+1})$. Then $i^*\mu \in S^i(m_j)$. Now cap with ζ_j, and we have

$$i^*\mu \cap \zeta_j = \sum n_\sigma < i^*\mu, \mathrm{fr}_i(\sigma) > \mathrm{bk}_{n-i}(\sigma).$$

On the other hand, we have

$$\mu \cap (\zeta_j + \chi_{j+1}) = \sum n_\sigma < \mu, \mathrm{fr}_i(\sigma) > \mathrm{bk}_{n-i}(\sigma) + \sum n_\tau < \mu, \mathrm{fr}_i(\tau) > \mathrm{bk}_{n-i}(\tau)$$

The second sum is in $S_{n-i}(X_j{+}1)$ and so equals zero in $H_{n-i}(M_j, \partial M_j) = H_{n-i}(M_{j+1}, X_{j+1})$. Thus, we are left with,

$$\mu \cap (\zeta_j + \chi_{j+1}) = \sum n_\sigma < \mu, \mathrm{fr}_i(\sigma) > \mathrm{bk}_{n-i}(\sigma) \in S_{n-i}(M_j)$$

and the diagram commutes. Note that although this only proves Lefschetz duality for $i \neq k-1, k$, the diagram commutes for all i.

Now, for $i = k-1, k$ we have,

$$\begin{tikzcd}
 & & & \mathbb{Z} \arrow[d, equal] & \\
0 \arrow[r] & H_{n-k+1}(M_{j+1}, \partial M_{j+1}) \arrow[r, "i^*"] & H_{n-k+1}(M_j, \partial M_j) \arrow[r] & H_{n-k}(D^{n-k}, \partial D^{n-k}) \arrow[r] & \cdots \\
0 \arrow[r] & H^{k-1}(M_{j+1}) \arrow[r] \arrow[u, "\cap[M_{j+1}, \partial M_{j+1}]"'] & H^{k-1}(M_j) \arrow[r] \arrow[u, "\cap[M_j, \partial M_j]"', "\cong"] & H^k(D^k, \partial D^k) \arrow[r] \arrow[u, "\cap[X_{j+1}, \partial X_{j+1}]"'] \arrow[d, equal] & \cdots \\
 & & & \mathbb{Z} &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n-k}(M_{j+1}, \partial M_{j+1}) \arrow[r] & H_{n-k}(M_j, \partial M_j) \arrow[r] & 0 \\
\cdots \arrow[r] & H^k(M_{j+1}) \arrow[r] \arrow[u, "\cap[M_{j+1}, \partial M_{j+1}]"'] & H^k(M_j) \arrow[r] \arrow[u, "\cap[M_j, \partial M_j]"', "\cong"] & 0
\end{tikzcd}$$

So, we need to show that the map

$$\begin{tikzcd}
H_{n-k}(D^{n-k}, \partial D^{n-k}) \arrow[r, equal] & H_{n-k}(X_{j+1}, \partial M_{j+1}) \\
H^k(D^k, \partial D^k) \arrow[r, equal] \arrow[u, "\cap[X_{j+1}, \partial X_{j+1}]"', "\cong ?"] & H^k(X_{j+1}, \partial M_j) \arrow[u, "\cap[X_{j+1}, \partial X_{j+1}]"', "\cong ?"]
\end{tikzcd}$$

is an isomorphism $\square$

$\square$

7.3.1 More cup and cap product

If M is a closed oriented n-manifold, then Poincaré duality tells us we have a fundamental class $[M] \in H_n(M)$ and

$$\cap[M] : H^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

for all k. Consider the composition,

$$H^k(M) \otimes H^{n-k}(M) \xrightarrow{\cup} H^n(M) \xrightarrow{<\ ,[M]>} \mathbb{Z}$$

This gives a bilinear pairing, $H^k(M) \otimes H^{n-k}(M) \to \mathbb{Z}$, given by,

$$\alpha \otimes \beta \mapsto < \alpha \cup \beta, [M] > = < \alpha, \beta \cap [M] > = < \alpha, \mathrm{PD}(\beta) >$$

where $\mathrm{PD}(\beta)$ denotes the Poincaré dual of β. So, another way to write this pairing is,

$$H^k(M) \otimes H^{n-k}(M) \xrightarrow{1 \times \mathrm{PD}} H^k(M) \otimes H_k(M) \xrightarrow{<\ ,\ >} \mathbb{Z}$$

If we just consider the torsion part $\operatorname{Tor} H^k(M) \subset H^k(M)$ we see that the map $\operatorname{Tor} H^k(M) \otimes H^{n-k}(M) \to \mathbb{Z}$ is zero. To see this, suppose $\alpha \in \operatorname{Tor} H^k(M)$ and $l\alpha = 0$. Then

$$l(\alpha \otimes \beta) = (l\alpha \otimes \beta) = (0 \otimes \beta) \mapsto 0,$$

and thus $(\alpha \otimes \beta) \mapsto 0$. Similarly, $H^k(M) \otimes \operatorname{Tor} H^{n-k}(M) \to \mathbb{Z}$ is zero, so we think of this pairing as,

$$\frac{H^k(M)}{\operatorname{Tor} H^k(M)} \otimes \frac{H^{n-k}(M)}{\operatorname{Tor} H^{n-k}(M)} \to \mathbb{Z}.$$

Theorem 7.3.10. *This is a perfect pairing.*

If A, B are free abelian groups then $\bullet : A \otimes B \to \mathbb{Z}$ is a perfect pairing if its adjoint, $A \to \mathrm{Hom}(B, \mathbb{Z}) = B^*$ is an isomorphism. Another way to say this is, choosing bases for A, B, $\bullet$ is represented by a square matrix and it is a perfect pairing iff the determinant of this matrix is ± 1.

Proof. Our pairing is now,

$$\frac{H^k(M)}{\operatorname{Tor} H^k(M)} \otimes \frac{H^{n-k}(M)}{\operatorname{Tor} H^{n-k}(M)} \xrightarrow{1 \otimes \mathrm{PD}} \frac{H^k(M)}{\operatorname{Tor} H^k(M)} \otimes \frac{H_k(M)}{\operatorname{Tor} H_k(M)} \xrightarrow{<\ ,\ >} \mathbb{Z}$$

Poincaré duality tells us the first map in this composition is an isomorphism, so we need only to check that the pairing,

$$\frac{H^k(M)}{\operatorname{Tor} H^k(M)} \otimes \frac{H_k(M)}{\operatorname{Tor} H_k(M)} \xrightarrow{<\ ,\ >} \mathbb{Z}$$

is perfect. This is exactly what the universal coefficent theorem says. $\square$

What about the symmetry of this pairing?

Theorem 7.3.11. *$\cup : H^*(X) \otimes H^*(X) \to H^*(X)$ satisfies $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$.*

This should be a suprise to us. It certainly does not hold on the cochain level. In fact, we have the following theorem,

Theorem 7.3.12. *There cannot be a natural cochain cup product that is associative, graded cummutative and satisfies the Leibniz rule.*

The most interesting case of this pairing is when $k = n - k$ i.e. $n = 2k$. Then we have,

$$i : \frac{H^k(M^{2k}))}{\operatorname{Tor}(H^k(M^{2k}))} \otimes \frac{H^k(M^{2k}))}{\operatorname{Tor}(H^k(M^{2k}))} \to \mathbb{Z}$$

The pairing is perfect, and symmetric for k even and skew symmetric for k odd. If we take a basis $\{e_i\}$ for $H^k(M)/\operatorname{Tor}(H^k(M))$ then i is represented by a matrix $A_{lm} = i(e_l, e_m)$. This matrix has the property $A_{lm} = A_{ml}$ if k is even, and $A_{lm} = -A_{ml}$ if k is odd. The determinant of A_{lm} is ± 1. In the symmetric case we are considering quadratic forms and in the skew symmetric case, we are studying symplectic forms. The algebraic classification of these matrices correspond to the classification of these pairings up to isomorphism. These provide algebraic invariants of M (and an orientation).

Theorem 7.3.13. *A perfect skew symmetric pairing is isomorphic to a direct sum $\bigoplus_{i=0}^{n} H$ for $n \geq 0$, where*

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In particular, the rank of the free abelian group is even and is a complete invariant of the pairing up to isomorphism.

Corollary 7.3.14. *If M is a closed, oriented $4k+2$ manifold, then the rank of $H^{2k+1}(M)$ is even and $\mathrm{rk} H^{2k+1-r}(M) = \mathrm{rk} H^{2k+1+r}(M)$ for all $r > 0$.*

Corollary 7.3.15. *If M is a closed, oriented $4k+2$ manifold then the Euler characteristic of M is even.*

Example 7.3.16. To see that orientation is neccesary, notice that the Euler characteristic of $\mathbb{R}P^2$ is 1.

Corollary 7.3.17. *If M is a closed, oriented $2k+1$ manifold then the Euler characteristic of M is 0.*

Remark 7.3.18. This is also true for non-orientable manifolds, as can be shown using a $\mathbb{Z}/2$ formulation of Poincaré duality.

Now we turn to perfect symmetric pairings. These pairings are classified by three invariants.

- *rank*: The rank of the free abelian group.
- *signature*: Tensor with $\mathbb{R}$ and then diagonalize. Then the signature is the number of plus 1's on the diagonal minus the number of -1's.
- *parity*: A pairing is even if every diagonal entry is even, and odd otherwise.

Note that rank and signature are $\mathbb{R}$ invariants of the pairing, while parity is an integral invariant.

Theorem 7.3.19. *Let $(L, <,>)$ be a perfect symmetric pairing. If L is not positive defninite or negative definite, then the isomorphism class of $(L, <,>)$ is determined by the rank, signature and parity of the pairing.*

We remark without proof that the folowing relationships hold between these invariants:

- rk $\equiv$ signature mod 2.
- $|$signature$| \leq$ rk.
- If the parity is even then sign $\equiv 0$ mod 8.

Suppose that (L, Q) is a perfect symmetric pairing. Then the adjoint of Q identifies L with its dual L^* by $l \mapsto Q(l, \cdot) : L \to \mathbb{Z}$. So if we have another lattice M and a map $f : M \to L$, we can use this identification to think of the dual map $f^* : L^* \to M^*$ as a map $f^* : L \to M^*$.

Lemma 7.3.20. *Suppose that (L, Q) is a perfect symmetric pairing, $f^* \circ f = 0$ and* $\mathrm{Ker}\, f^*/\mathrm{Im} f$ *is torsion, then the signature of (L, Q) is 0.*

Theorem 7.3.21. *Let W be a compact, oriented $4n+1$ manifold. Let $M = \partial W$ be a closed oriented $4n$ manifold. Then the signature of M is zero.*

Exercise 7.3.22. *Compute the signature of $\Sigma_g \times \Sigma_g$, where Σ_g is the closed surface of genus g.*

Exercise 7.3.23. *Show that* $\mathrm{sign}(M \# N) = \mathrm{sign}(M) + \mathrm{sign}(N)$.

Exercise 7.3.24. *If $\bar{X}$ is X with the opposite orientation, show that* $\mathrm{sign}(X \# \bar{X}) = 0$.

8 Differential Topology

In order to further develop the application of algebraic topology to manifolds, we will need to briefly study some important results from differential topology. Differential topology is a very interesting subject in its own right, but here we will only quickly cover the tools that will be neccesary for our applications. Our main goals will be first to define connections and covariant derivatives, so that we may describe the geodesic equation. We will then use the local existence of geodesics to define the exponential map and prove the tubular neighborhood theorem. Then we will return to algebraic topology, making use of the tubular neighborhood theorem to prove the Thom isomorphism theorem.

8.1 Connections

Consider the tangent bundle to a smooth manifold,

$$\begin{array}{c} TM \\ \Big\downarrow{\scriptstyle \pi} \\ M \end{array}$$

A *connection* on TM will be a choice of linear subspaces $H_{(x,v)} \subset T(TM)_{(x,v)}$. These subspaces will be the same dimension as the dimension of M and have the property that the differential $D\pi_{(x,v)} : T(TM)_{(x,v)} \to TM_x$ will be an isomorphism when restricted to these subspaces,

$$D\pi_{(x,v)} : H_{(x,v)} \xrightarrow{\cong} TM_x$$

We also require that the following two properties hold:

1. The $H_{(x,v)}$ vary smoothly with (x, v) i.e. the make a C^∞ distribution.
2. They are invariant under the vector space structure, i.e. $H_{(x,v)} + H_{(x,w)} = H_{(x,v+w)}$ and $r \cdot H_{(x,v)} = H_{(x,rv)}$ for all $r \in \mathbb{R}$.

By $H_{(x,v)} + H_{(x,w)} = H_{(x,v+w)}$ we mean, if we consider the map

$$+ : TM \times_M TM \to TM$$

given by

$$(x, v), (x, w)) \mapsto (x, v + w),$$

then $\{H_{(x,v)}, H_{(x,w)}\}$ defines a linear supspace of $TM \times_M TM$ of dimension equal to the dimension of M, and the differential $D(+)$ maps $\{(h, h')|(\pi_*(h) = \pi_*(h')\} \mapsto H_{(x,v+w)}$. Similarly, by $r \cdot H_{(x,v)} = H_{(x,rv)}$ for all $r \in \mathbb{R}$, we mean if we consider the map

$$r\cdot : TM \to TM$$

given by

$$(x, v) \mapsto (x, rv),$$

then the differential
$$D(r\cdot) : T(TM)_{(x,v)} \to T(TM)_{(x,v)}$$
takes $H_{(x,v)}$ to $H_{(x,rv)}$.

8.2 Covariant Derivatives

Choosing a connection on a manifold is equivalent to defining what is called a *covariant derivative*. Let VF denote the space of vector fields on M. Then a covariant derivative is a map,
$$\nabla : VF \otimes_{\mathbb{R}} VF \to VF.$$
Given two vector fields X and Y on M we write $\nabla_X(Y) \in VF$. We require this map to have the following three properties:

1. $\nabla_{r_1X_1+r_2X_2}(s_1Y_1 + s_2Y_2) = \sum r_i s_j \nabla_{X_i}(Y_j)$ for $r_i, s_j \in \mathbb{R}$.
2. $\nabla_{fX}(Y) = f\nabla_X(Y)$ for any $f \in C^\infty(M)$.
3. $\nabla_X(fY) = X(f) \cdot Y + f\nabla_X(Y)$

Claim 8.2.1. *$\nabla_X(Y)(p)$ depends only on $X(p)$ and the germ of Y at p.*

Proof. First, suppose that $\mathrm{germ}_p(Y) = \mathrm{germ}_p(Y')$. Then $Y - Y' = 0$ near p, so there exists a function f such that $f(Y - Y') = 0$ and $f' \equiv 1$ near p. Then $fY = fY'$ and $f \equiv 1$ near p. Applying our covariant derivative,
$$\nabla_X(fY)(p) = X(f)(p)Y + f\nabla_X(Y)(p) = \nabla_X(Y)(p)$$
since $X(f)Y(p) = 0$. Then the same computation shows $\nabla_X(fY')(p) = \nabla_X(Y')(p)$ and thus $\nabla_X(Y)(p) = \nabla_X(Y')(p)$.

Now, suppose that $X(p) = 0$. Take local coordinates $\{x^i\}$ centered at p so that $X = \sum f_i \frac{\partial}{\partial x^i}$ and $f_i(0) = 0$. Then
$$\nabla_X(Y)(p) = \sum f_i \nabla_{\frac{\partial}{\partial x_i}}(Y)(p) = \sum f_i(p)\nabla_{\frac{\partial}{\partial x_i}}(Y)(p) = 0$$
since $f_i(p) = 0$. □

Corollary 8.2.2. *Suppose that $U \subset M$ is an open subset. Then a connection ∇ on TM induces ∇^U on TU*

Proof. Suppose that X and Y are vector fields on U. We need to define $\nabla^U_X Y(p)$. Using a bump function argument, there exist vector fields $\hat{X}, \hat{Y}$ on M such that $\mathrm{germ}_p\hat{X} = \mathrm{germ}_p X$ and $\mathrm{germ}_p\hat{Y} = \mathrm{germ}_p Y$. Then define $\nabla^U_X Y(p) = \nabla_{\hat{X}}\hat{Y}(p)$. By the previous claim this is well defined. □

Now, let ∇ be given on TM and let $U \subset M$ be a coordinate patch with local coordinates $(x^1, \ldots, x^n)$. We define the *Christoffel symbols*, $\Gamma^k_{ij}(x^1, \ldots, x^n) \in C^\infty(U)$ by

$$\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) = \sum_{k=1}^{n} \Gamma^k_{ij}(x^1, \ldots, x^n)\frac{\partial}{\partial x^k}$$

Notice that these symbols are uniquely determined by the connection and these symbols determine the connection.

On a Riemannian manifold there is a best connection called the *Levi-Civta connection*. Recal that a Riemannian metric is a symmetric, bilinear, positive definite map,

$$<,>: TM \times_M TM \to \mathbb{R}$$

that varies smoothly with $x \in M$. In local coordinates, $(x^1, \ldots, x^n)$, the metric is given by a smoothly varying, symmetric, positive definite matrix,

$$\Big(g_{ij}(x^1, \ldots, x^n)\Big).$$

Suppose that M has a given Riemannian metric. If X, Y, Z are vector fields on M, then

$$X(< Y, Z >) =< \nabla_X, Z > + < Y, \nabla_X, Z >$$

so if we let $G(x) = g_{ij}(x)$, then

$$\frac{d}{dx}(Y^{\mathrm{tr}}GZ) = (\nabla_X(Y))^{\mathrm{tr}}GZ + Y^{\mathrm{tr}}\nabla_X(G)Z + Y^{\mathrm{tr}}G\nabla_X(Z).$$

Since this must hold for any vector fields X, Y, Z, we see that $\nabla_X G = 0$ i.e.

$$\nabla <,>= 0.$$

In local coordinates we have,

$$\frac{\partial}{\partial x^i} < \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} >=< \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} > + < \frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k} >$$

In terms of the Christoffel symbols, this is,

$$\frac{\partial}{\partial x^i}(g_{jk}) =< \sum_{l=1}^{n}\Gamma^l_{ij}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} > + < \frac{\partial}{\partial x^j}, \sum_{l=1}^{n}\Gamma^l_{ik}\frac{\partial}{\partial x^l} > .$$

Thus,

$$\frac{\partial g_{ik}}{\partial x^i} = \sum_{l=1}^{n}\Big(\Gamma^l_{ij}g_{lk} + \Gamma^l_{ik}g_{jl}\Big).$$

We also require that the Levi-Civta connection is torsion free, or symmetric. This means,

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

To see what this means in terms of the Christoffel symbols consider,

$$\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) - \nabla_{\frac{\partial}{\partial x^j}}(\frac{\partial}{\partial x^i}) = [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0.$$

Thus,

$$0 = \sum_{l=1}^{n} \Gamma_{ij}^{l} \frac{\partial}{\partial x^l} - \sum_{l=1}^{n} \Gamma_{ji}^{l} \frac{\partial}{\partial x^l},$$

which implies,

$$\Gamma_{ij}^{l} = \Gamma_{ji}^{l}$$

for all i, j, l.

Theorem 8.2.3. *On a Riemannian manifold there exists a unique torsin free metric connection. This is the Levi-Civta connection.*

Proof. Let $p \in M$ and let $(x^1, \ldots, x^n)$ be local coordinates centered at p such that $g_{ij}(p) = \delta_{ij}$. Then at p we have,

$$\frac{\partial g_{ij}}{\partial x^k}(p) = \Gamma_{ij}^{k} + \Gamma_{ik}^{j}.$$

Then a short computation shows that,

$$\frac{\partial g_{ij}}{\partial x^k}(p) - \frac{\partial g_{jk}}{\partial x^i}(p) + \frac{\partial g_{ki}}{\partial x^j}(p) = 2\Gamma_{ik}^{j}.$$

Thus,

$$\Gamma_{ik}^{j} = (1/2)\Big(\frac{\partial g_{ij}}{\partial x^k}(p) - \frac{\partial g_{jk}}{\partial x^i}(p) + \frac{\partial g_{ki}}{\partial x^j}(p)\Big).$$

□

8.3 Geodesics

We say that a path $\gamma : (-\epsilon, \epsilon) \to M$ is a *geodesic* if $\gamma'(t) \in TM_{\gamma(t)}$ is a parallel family i.e. $\nabla_{\gamma'(t)}\gamma'(t) = 0$. We need to explain what we mean by this since $\gamma'(t)$is not a vector field on M. Let $\tau(t) \in TM_{\gamma(t)}$. We want to define $\nabla_{\gamma'(t)}\tau(t) \in TM_{\gamma(t)}$. Suppose we have local coordinates $(x^1, \ldots, x^n)$ so that

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$$

$$\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$$
$$\tau(t) = (\tau_1(t), \dots, \tau_n(t))$$

Then we want,

$$\begin{aligned}\nabla_{\sum \gamma_i'(t)\frac{\partial}{\partial x^i}}(\tau(t)) &= \sum \gamma_i'(t)\nabla_{\frac{\partial}{\partial x^i}}\left(\tau_1(t)\frac{\partial}{\partial x^1}, \dots, \tau_n(t)\frac{\partial}{\partial x^n}\right)\\ &= \sum_{i=1}^n \gamma_i'(t)\bigg(\sum_{j=1}^n \frac{\partial \tau_j}{\partial x^i}\frac{\partial}{\partial x^j} + \tau_j \Gamma_{ij}^k \frac{\partial}{\partial x^k}\bigg).\end{aligned}$$

Now,

$$\sum_{i=1}^n \gamma_i'(t)\frac{\partial \tau_j(t)}{\partial x^i} = \frac{\partial \tau_j}{\partial t}$$

And so,

$$\sum_{i=1}^n \gamma_i'(t)\bigg(\sum_{j=1}^n \frac{\partial \tau_j}{\partial x^i}\frac{\partial}{\partial x^j} + \tau_j \Gamma_{ij}^k \frac{\partial}{\partial x^k}\bigg) = \sum_{j=1}^n \tau_j'(t)\frac{\partial}{\partial x^j} + \sum_{i,j,k=1}^n \gamma_i'(t)\tau_j(t)\Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

And we define this to be, $\nabla_{\gamma'(t)}\tau(t)$. Now it makes sense to require that

$$\nabla_{\gamma'(t)}\gamma'(t) = 0.$$

In local coordinates this becomes,

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{j=1}^n \gamma_j''(t)\frac{\partial}{\partial x^j} + \sum_{i,j,k=1}^n \gamma_i'(t)\gamma_j'(t)\Gamma_{ij}^k\frac{\partial}{\partial x^k} = 0.$$

For a fixed k this gives,

$$\gamma_k''(t) + \sum_{i,j=1}^n \Gamma_{ij}^k(\gamma(t))\gamma_i'(t)\gamma_j'(t),$$

and so this gives us n second order ordinary differential equations for $\gamma_1(t), \dots, \gamma_n(t)$.

Theorem 8.3.1. *Given $p \in M$ and $\tau \in TM_p$, there exists an $\epsilon > 0$ and a geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = \tau$. Furthermore, any two such geodesics agree on their common domain of definition.*

Furthermore, if X is a smooth manifold and $\phi : X \to TM$ is a smooth map, then there exists $\epsilon : X \to (0, \infty)$ so that if we let $U_\epsilon = \{(x,t) | -\epsilon(x) < t < \epsilon(t)\} \subset X \times \mathbb{R}$ there exists $\psi : U_\epsilon \to M$ such that $\psi|_{X\times\{0\}} = \pi \circ \phi$, $\frac{\partial \psi}{\partial t}|_{X\times\{0\}} = \phi$ and $\psi|_{X\times(-\epsilon(x),\epsilon(x))}$ is a geodesic.

Now we can define the exponential map. Let $X = TM_p$. Let $N(0)$ be a neighborhood of 0 in X. Then we have a map, $\phi : N(0) \to M$ given by $\phi(x) = \gamma_x(1)$ i.e. $\psi|_{N(0)\times\{1\}}$, and $D\phi_p = \mathrm{Id}_{TM_p}$.

8.3.1 The Tubular Neighborhood Theorem

Let X be an n-dimensional compact smoothly embedded manifold in an N-dimensional smooth manifold M. Impose a Riemannian metric on M. Then we have the normal bundle $\nu_{X\subset M} = TM|_X/TX$. The Riemannian metric provides an embedding of the normal bundle back into the tangent bundle of M restricted to X, $\nu_{X\subset M} \hookrightarrow TM|_X$ as the orthogonal complement to $TX \subset TM|_X$.

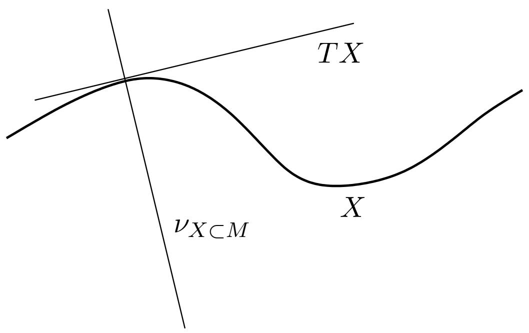

Figure 16: The normal bundle is identified with the orthogonal complement to $TX \subset TM|_X$

So now the normal bundle sits in $TM|_X$ as a subbundle, so we have

$$\begin{array}{ccc} \nu_{X\subset M} & \xrightarrow{j} & TM \\ \downarrow & & \downarrow \\ X & \longrightarrow & M \end{array}$$

Now, we want to define a map from the normal bundle to M by $(x, v) \mapsto \gamma_{(x,v)}(1)$, where $\gamma_{(x,v)}$ is the geodesic in M such that $\gamma_{(x,v)}(0) = x$ and $\gamma'_{(x,v)}(0) = v$. If $\gamma_{(x,v)}(1)$ is defined for some point (x, v) in the normal bundle, then $\gamma_{(x',v')}(1)$ is defined for all (x', v') sufficently close to (x, v). Notice that $\gamma_{x,0)}(1)$ is defined for all $x \in X$. It is simply $\gamma_{x,0)}(1) = x$. This implies there exists some neighborhood U_x of $x \in X$ and an $\epsilon_x > 0$ such that $\gamma_{(u,v)}(1)$ is defined for all $u \in U_x$ and v with $|v| < \epsilon_x$. Since X is compact there exists some $\epsilon > 0$ such that $\gamma_{(x,v)}(1)$ is defined for all points $(x, v) \in \nu_{X\subset M}$ where $|v| < \epsilon$. Let

$$\nu_{X\subset M,\epsilon} = \{(x, v) \in \nu_{X\subset M} |\ |v| < \epsilon\}.$$

Then we have a smooth map from this ϵ tube about the zero section of the normal bundle to M,

$$exp : \nu_{X\subset M,\epsilon} \to M$$
$$(x, v) \mapsto \gamma_{(x,v)}(1).$$

Now, lets examine $Dexp_(x, 0)(0, v)$. This is a map,

$$T\nu_{X\subset M}(x, 0) = TX_x \oplus (\nu_{X\subset M})(x) \to TM_x = TX_x \oplus (\nu_{X\subset M})(x).$$

If we consider how this exponential map changes as we vary x in X along the zero section of $\nu_{X\subset M}$ we see that $Dexp_{(x,0)}(0,v)$, is in fact the identity. So,

$$exp(x,tv) = \gamma_{(x,v)}(t),$$

and,

$$Dexp_{(x,0)}(0,v) = \gamma'_{(x,v)}(0) = v.$$

Then by the compactness of X and the inverse function theorem there is some $0 < \epsilon' < \epsilon$ such that $\exp : \nu_{X\subset M,\epsilon'} \to M$ is a local diffeomorphism onto an open set. To get a diffeomorphism then we just need to show that the map is one to one. In order to do this we may need to shrink ϵ'.

Lemma 8.3.2. *For possibly smaller ϵ', the map $exp : \nu_{X\subset M,\epsilon'} \to M$ is one-to-one and hence a diffeomorphism onto its image, which is an open neighborhood of $X \subset M$.*

Proof. Suppose that no such smaller ϵ' exists. Then there are sequences $\{(x_n,v_n),(y_n,w_n)\}$ such that $exp : (x_n,v_n) = exp(y_n,w_n)$ for all (x_n,v_n) and (y_n,w_n), with $(x_n,v_n) \neq (y_n,w_n)$ for any n and $|v_n|,|w_n| \to 0$ as $n \to \infty$. Since X is compact, we can pass to a subsequence and assume that $x_n \to x$ and $y_n \to y$. Thus, $(x_n,v_n) \to (x,0)$ and $(y_n,w_n) \to (y,0)$. By continuity $exp(x,0) = exp(y,0)$, which implies that $x = y$ and therefore both $(x_n,v_n) \to (x,0)$ and $(y_n,w_n) \to (x,0)$. But this contradicts the fact that exp is a local diffeomorphism. □

In particular, this shows that if $X \subset M$ is a codimension 1 submanifold which is locally 2-sided in M i.e. $\nu_{X\subset M}$ is orientable and therefore trivial, then a neighborhood of X in M is diffeomorphic to $X \times (-\epsilon,\epsilon)$.

8.4 The Thom Isomorphism Theorem

Let M be a smooth manifold and

$$\begin{array}{c} V \\ \big\downarrow{\scriptstyle \pi} \\ M \end{array}$$

be a smooth dimension n vector bundle. Suppose that V is orientable as a vector bundle. By this we mean that the line bundle $\Lambda^n V = \mathcal{L} \to M$ is orientable ($\Leftrightarrow$ $\mathcal{L}$ is trivial) or equivalently, there is a local trivialisation of V such that all of the transition functions have positive determinant. For an n-dim vector space V, an orientation is equivalent to a generating class $U_V \in H^n(V, V\setminus\{0\})$. If $\sigma : \Delta^n \hookrightarrow V$ is a linear embedding, $0 \in \sigma(\text{int}\Delta^n)$ and σ is orientation preserving, then $< U_V, [\sigma] >= +1$.

Theorem 8.4.1. Thom Isomorphism Theorem *Let $\pi : V^n \to M$ be an oriented vector bundle. Then there exists a $U \in H^n(V, V\setminus\{0\}section)$ called the Thom class such that*

1. *$U|_{V_x} \in H^n(V_x, V_x\setminus\{0\})$ is the cohomology class determined by the orientation for every $x \in M$.*

2. $\cup U : H^k(V) = H^k(M) \to H^{k+n}(V, V \setminus \{0\} section)$ *is an isomorphism for all* K.

To shorten our notation, from now on for a vector bundle V, we will write $H^*(V, \setminus\{0\}section)$ for $H^*(V, V \setminus \{0\}section)$.

Remark 8.4.2.
- Suppose that the vector bundle $V \to M$ has a metric. Let V_ϵ be the subbundle of balls of radius ϵ. Then $U|_{V_\epsilon} \in H^n(V_\epsilon, \setminus\{0\}section)$ satisfies statement 2.
- Let $\bar{V}_\epsilon$ be the subbundle of closed balls of radius ϵ and let $\partial\bar{V}_\epsilon$ be the bundle of spheres of radius ϵ. Then there exists a class $U \in H^n(\bar{V}_\epsilon, \partial\bar{V}_\epsilon)$ such that $\cup U : H^k(\bar{V}_\epsilon) = H^k(M) \to H^{k+n}(\bar{V}_\epsilon, \partial\bar{V}_\epsilon)$ is an isomorphism for all k.

One important application of the Thom isomorphism theorem is in computing cohomology classes which are Poincaré dual to homology classes represented by embedded submanifolds. Suppose that M is a closed oriented n-manifold and $i : X^k \hookrightarrow M$ is a closed oriented submanifold. Then we have $i_*[X] \in H_k(M)$. Then the Poincaré dual to $i_*[X]$, $PD^{-1}(i_*[X]) \in H^{n-k}(M)$ is given as follows. First, notice that the normal bundle $\nu_{X \subset M}$ is oriented since both TX and TM are oriented. For every $\epsilon > 0$ we have the Thom class $U \in H^{n-k}(\nu_{X\subset M,\epsilon}, \setminus\{0\}section)$. If we take ϵ sufficently small then the tubular neighborhood theorem identifies $\nu_{X\subset M,\epsilon}$ with a tubular neighborhood of $X \subset M$ via the exp map. Now we push the class forward,

$$(exp^{-1})^*U \in H^{n-k}(\text{nbhd}(X \subset M), \text{nbhd}(X \subset M) \setminus X) = H^{n-k}(M, M \setminus X) \to H^{n-k}(M),$$

and this is the Poincaré dual, $PD^{-1}(i_*[X]) \in H^{n-k}(M)$.

Corollary 8.4.3. *Suppose that X^k and Y^{n-k} are closed oriented smooth submanifolds of M^n a closed oriented smooth manifold. Suppose that X and Y meet transversally i.e. for every $p \in X \cap Y$, TX_p and TY_p are complementary subspaces of TM_p. Then $X \cap Y$ is a finite set of points and to each point $p \in X \cap Y$ we have a sign, $\epsilon(p) = \pm 1$, given by*

$$\sigma_{TX_p} \oplus \sigma_{TY_p} = \sigma_{TM_p}\epsilon(p).$$

Then,

$$\sum_{p \in X \cap Y} \epsilon(p) =< PD^{-1}[X], [Y] >=< PD^{-1}[X] \cup PD^{-1}[Y], [M] > .$$

Example 8.4.4. Consider $\mathbb{C}P^2$ with its complex orientation. We claim theat $x = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$ generates. We take two representatives for x, $L_1 = [z_0, z_1, 0] \in \mathbb{C}P^2$ and $L_2 = [0, z_1, z_2] \in \mathbb{C}P^2$. Then $L_1 \cap L_2 = \{[0, z_1, 0]\} = \{p\} \in \mathbb{C}P^2$. Let $\zeta_0 = z_0/z_1$ and $\zeta_2 = z_/z_0$. Then these are local coordinates near this point, and in these coordinates $L_1 = \{\zeta_0 = 0\}$ and $L_2 = \{\zeta_2 = 0\}$, so clearly they intersect transversally. Thus, $< PD^{-1}(x), x >=< PD^{-1}(x) \cup PD^{-1}(x), [\mathbb{C}P^2] >= 1$.

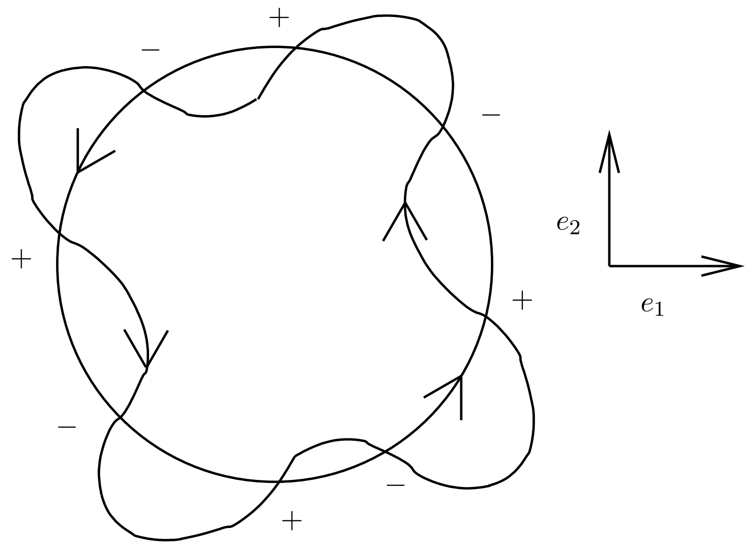

Figure 17: The algebraic intersection of two 1-submanifolds in $\mathbb{R}^2$

Example 8.4.5. Consider $S^2 \times S^2$. We claim that the map $H_2(S^2 \times S^2) \otimes H_2(S^2 \times S^2) \to \mathbb{Z}$, given by $x \otimes y \mapsto < PD^{-1}(x), y >$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example 8.4.6. The surface of genus 2, Σ_2, with generators $\{a_1, b_1, a_2, b_2\}$ for $H_1(\Sigma_2)$ as pictured. Then the map $H_1(\Sigma_2) \otimes H_1(\Sigma_2) \to \mathbb{Z}$ given by $x \otimes y \mapsto < PD^{-1}(x), y >$ is given by the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

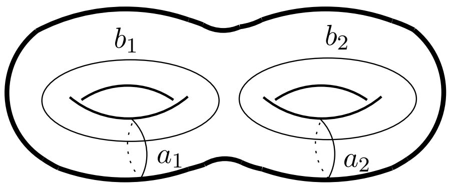

Figure 18: Generators for $H_1(\Sigma_2)$

Now, we prove the Thom Isomorphism Theorem.

Proof. Let $f : M \to \mathbb{R}$ be a Morse function with critical values $c_1 < \cdots < c_k$ such that $f^{-1}(c_i)$ has a unique critical point p_i for each i. Take $a_0 < c_1 < a_1 < c_2 < \cdots < c_k < a_k$.

Let $M_{(-\infty,a_r]} = f^{-1}(-\infty, a_r]$. Then $M_{(-\infty,a_r]}$ is a smooth submanifold of M with boundary $\partial M_{(-\infty,a_r]} = M_{a_r} = f^{-1}(a_r)$. We will prove by induction that the result holds for $V|_{M_{-\infty,a_r]}}$.

Lemma 8.4.7. *Let $\pi : V^n \to Y \times I$ be a smooth vector bundle. Then there is a vector bundle isomorphism covering the identity on the base:*

$$\begin{array}{ccc} V|_{Y\times\{0\}} \times I & \longrightarrow & V \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ Y \times I & = & Y \times I \end{array}$$

Proof.

Claim 8.4.8. *Let $W \to Y$ be a smooth vector bundle. Then there is a smooth distribution $\{H_w \subset TW_w\}_{w \in W}$ such that,*

- $\pi_* : H_w \to TY_{\pi(w)}$ *is an isomorphism.*
- $H_{rw} = rH_w$.
- $H_{w_1+w_2} = H_{w_1} + H_{w_2}$ *if* $\pi(w_1) = \pi(w_2)$.

Proof. Locally in Y distributions exist, since W is locally trivial, given $y \in Y$ there is an open neighborhood $U \subset Y$ of y and a vector bundle isomorphism from $W|_U \to U \times V_0$. Let $H_{(U,V_0)} = TU_{(U,V_0)}$, the trivial connection on the trivial bundle. Now, cover Y by open sets U_α with distributions $\mathcal{H}^\alpha = \{H_x^\alpha\}$ on $W|_{U_\alpha}$. Let λ_α be a partition of unity subordinate to this cover. We want to form $\sum \lambda_\alpha \mathcal{H}^\alpha = \mathcal{H}$. We want to define $\sum \lambda_{\alpha_i}(y) H_W^{\alpha_i} = H_W$. We have, there exists a unique $p_w^{\alpha_i} : TW_w \to W_y$ for $w \in \pi^{-1}(y)$ such that the kernel of $p_w^{\alpha_i} = H_w^{\alpha_i}$ and $p_w^{\alpha_i}|_{W_y}$ is the identity on W_y. Now,

$$p_w = \sum \lambda_{\alpha_i}(y) p_w^{\alpha_i} : TW_w \to W_y.$$

This map is the identity restricted to W_y, so $\operatorname{Ker} p_w = H_w$ is complementary to W_y. So, $\mathcal{H} = \sum \lambda_\alpha \mathcal{H}^\alpha$ is a connection on $W \to Y$. $\square$

Now, we apply this to $V \to Y \times I$. So, we have a connection $\mathcal{H}$ on this vector bundle. We claim that for each $v_0 \in V|_{Y\times\{0\}}$ there exists a unique $\gamma : [0,1] \to V$ so that $\gamma(0) = v_0$ and $\pi(\gamma(t)) = (\pi(v_0), t)$ and $\gamma'(t) \in H_{\gamma(t)}$. Restrict V to $V|_{\{y\}\times I}$. The connection also restricts to give a connection on this bundle, $\bar{\mathcal{H}}_w = (d\pi)^{-1}(T_{\{y\}\times I})$ =a line in $(TW|_{\{y\}\times I})_w$. Now, trivialize the bundle, $V|_{\{y\}\times I} = V_0 \times (\{y\} \times I)$. Then H_w is just a graph, $(\frac{d}{dt}, L_w(\frac{d}{dt}))$ where $L_w : \mathbb{R} \to V_0$ is a smoothly varying linear map such that $L_{rw} = rL_w$ and $L_{w_1+w_2} = L_{w_1} + L_{w_2}$. Impose a metric on V_0. Then $|L_w(\frac{d}{dt})| \geq 0$. Let $K = \max\limits_{|w|=1} |L_w(\frac{d}{dt})|$. Then we claim $|\gamma'(t)| \leq K|\gamma(t)|$ for a horizontal $\gamma(t)$, and this implies that $|\gamma(t)| \leq ce^{Kt}$ for some constant c. So, we have solutions to the differential equation for all time and we get,

$$\begin{array}{ccc} V|_{Y\times\{0\}} \times I & \xrightarrow{\Phi} & V \\ \downarrow & & \downarrow \\ Y \times I & = & Y \times I \end{array}$$

and Φ is a linear isomorphism on each fiber. $\square$

Corollary 8.4.9. *Suppose X is a smoothly orientable manifold and $V \to X$ is a smooth vector bundle, then V is trivial.*

Proof. Let $H : Y \times I \to Y$ be a smooth contraction. Take $H^*V \to Y$. Then $H^*V|Y \times \{0\} = V$ and $H^*V|_{Y\times\{1\}} = Y \times V_y$. Thus, V is vector bundle isomorphic to $Y \times V_y$. $\square$

Similarly, if $Y \subset X$ and X smoothly deforms to Y then any smooth vector bundle $V \to X$ smoothly deforms linearly in the fibers to a vector bundle $V|_Y \to Y$.

Now, the Thom isomorphism theorem holds for $X = \{p\}$. We have $V^n \to \{p\}$, and the Thom class $U \in H^n(V, V \setminus \{0\})$ is the generator associated to the orientation of V. Also,

$$\cup C : H^k(V) \to H^{n+k}(V, V \setminus \{0\})$$

is an isomorphism for all k. The theorem is also true for $X = D^n$. Since the disk is contractible, $V \to D^n$ is vector bundle isomorphic to $V_0 \times D^n \to D^n$, and

$$\begin{array}{ccc} H^*(V, \setminus\{0\}section) & \xrightarrow{\cong} & H^*(V_0, V_0 \setminus \{0\}) \\ {\scriptstyle \cup U}\downarrow & & {\scriptstyle \cup U}\downarrow \\ H^*(V) & = & H^*(D^n) \end{array}$$

Claim 8.4.10. *The Thom isomorphism theorem holds for $X = S^k$.*

Proof. The proof is by induction on k. For $k = 0$, $S^0 = D^0 \coprod D^0$, and the theorem has been shown. Now, $S^k = D^k_+ \cup_{S^{k-1}} D^k_-$. By induction we have,

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
H^{n+l}(V, \backslash\{0\}section) & \longrightarrow & H^l(S^k) \\
\downarrow & & \downarrow \\
H^{n+l}(V|_{D_+}, \backslash\{0\}section) \oplus H^{n+l}(V|_{D_-}, \backslash\{0\}section) & \xrightarrow[\cong]{(\cup U_+, \cup U_-)} & H^l(D^k_+) \oplus H^l(D^k_-) \\
\downarrow & & \downarrow \\
H^{n+l}(V|_{S^{k-1}}) & \xrightarrow[\cong]{\cup U'} & H^l(S^{k-1}) \\
\downarrow & & \downarrow
\end{array}$$

So there exists a Thom class $U \in H^n(V, \backslash\{0\}section)$ so that,

$$\cup U : H^{n+l}(V, \backslash\{0\}section) \to H^l(S^k)$$

makes the diagram commute, and a five lemma argument shows that the map is an isomorphism. $\square$

Now, we have proven the theorem for X a disk and X a sphere. First, consider $X_{(-\infty,a_1]}$ This is just a disk, so the theorem has already been proven in this case. Suppose that the theorem is true for $X_{(-\infty,a_{j-1}]} \subset X_{(-\infty,a_j]}$. So, we have a Thom class $U_{j-1} \in H^n(V|_{X_{(-\infty,a_{j-1}]}}, \backslash\{0\}section)$. Recall that $X_{(-\infty,a_j]}$ deformation retracts to $X_{(-\infty,a_{j-1}]} \cup_{S^{r-1}} D^r$, so,

$$H^*(V|_{X_{(-\infty,a_j]}}, \backslash\{0\}section) \cong H^*(V|_{X_{(-\infty,a_{j-1}]} \cup D^r}, \backslash\{0\}section).$$

We have, $V|_{X_{(-\infty,a_{j-1}]} \cup D^r} = V|_{X_{(-\infty,a_{j-1}]}} \cup_{V|_{S^{r-1}}} V|_{D^r}$. We obtain the sequence,

$$0 \to H^n(V|_{X_{(-\infty,a_{j-1}]} \cup D^r}, \backslash\{0\}section) \to H^n(V|_{(-\infty,a_{j-1}]}, \backslash\{0\}section) \oplus H^n(V|_{D^r}, \backslash\{0\}section) \to$$

$$H^n(V|_{S^{r-1}}, \backslash\{0\}section) \to \cdots$$

This shows that there exists a class $U_j \in H^n(V|_{X_{(-\infty,a_{j-1}]} \cup D^r}, \backslash\{0\}section)$, and another 5-lemma argument shows that this is the desired Thom class. $\square$

Remark 8.4.11. This theorem is true for any topological vector bundle over a paracompact base. The general proof uses local triviality plus paracompactness to establish the homotopy result for vector bundles. Then a spectral sequence argument replaces the iunductive argument given here.

Corollary 8.4.12. *Let M^n be a smooth manifold, $X^k \subset M$ a closed smooth submanifold with oriented normal bundle. Then there exists a class $U \in H^{n-k}(M, M \setminus X)$ such that if $i : D^{n-k} \hookrightarrow M$ is a smooth map transverse to X (i.e. $TM_x = TX_x \oplus \operatorname{Im}(Di)_0$) mapping 0 to X and $D^{n-k} \setminus \{0\}$ to $M \setminus X$, then $< U, i_*[D, \partial D] >= \pm 1$ depending on the orientation of D^{n-k} versus the normal orientation of X.*

Corollary 8.4.13. *Let M^n be a smooth manifold, $X^k \subset M$ a closed smooth submanifold with oriented normal bundle. Suppose that $f : Y^{n-k} \to M$ is a smooth map, where Y is a closed oriented smooth manifold. Let $U \in H^{n-k}(M, M \setminus X)$ give the normal orientation for X. Suppose that f is transverse to X. Then $f^{-1}(X)$ is a finite set of points, and at each point we an compare the orientation of TY_y with the orientation of $(\nu_{X \subset M})_y$. Let $\epsilon(y) = \pm 1$ depending on wether or not these orientations agree or disagree. Then*

$$< U, [Y] >= \sum_{y \in X \cap Y} \epsilon(y)$$

Theorem 8.4.14. *Suppose that $X^k \subset M^n$ are closed oriented manifolds. Then $\nu_{X \subset M}$ has an induced orientation so that $o(X) \oplus o(\nu_{X \subset M}) = o(M)$. Let $U \in H^{n-k}(M, M \setminus X)$ be the image of the Thom class. Then the image of U in $H^{n-k}(M)$ is Poincaré dual to $[X]$*

Notice that U has a cocycle representative supported in an arbitrarily small neighborhood of X.

Proof. We need to show that for any $\alpha \in H^k(M)$,

$$\begin{aligned} < \alpha, [X] > &=< \alpha \cup U, [M] > \\ &=< \alpha, U \cap [M] > \\ &=< \alpha, PD(U) > . \end{aligned}$$

By construction U has a cocyle representative suported in a tubular neighborhood W of X. Thus, $U \cap M$ is a cycle of dimension k supported in W. W is a tubular neighborhood of X, so the inclusion $X \hookrightarrow W$ induces, $H_*(W) = H_*(X)$, and therefore $H^k(W) = \mathbb{Z}$ and $[X]$ is a generator. So, $U \cap [M] = t[X]$ for some $t \in \mathbb{Z}$. Fix a point $x \in X$ and a cocylce $\alpha \in S^k(X, X \setminus \{x\})$ such that $< \pi^*\alpha, [X] >= 1$. It suffices to compute

$$\begin{aligned} < \pi^*\alpha \cup U, [M] > &=< \pi^*\alpha, U \cap [M] > \\ &=< \pi^*\alpha, t[X] > \\ &= t < \pi^*\alpha, [X] > \\ &= t. \end{aligned}$$

□

Thus, $PD^{-1}[X]$ is a cohomology class with a cocycle representative supported near X and computing transverse intersection numbers with X.

A Category Theory

The language of category theory was introduced in the 1940's by S. Eilenberg and S. MacLane. It is a language well-suited to describe the tools of algebraic topology. In fact, Eilenberg and MacLane first discovered categories and functors while studying universal coefficent theorems in Čech cohomology. Although it has developed into a research topic in its own right, we will only make use of the very basic terminology.

A.1 Categories

A *category*, $\mathcal{C}$ consists of a collection of *objects* and for each oredered pair of objects A, B a set of *morphisms* between those objects, denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$ or simply $\mathrm{Hom}(A, B)$ if $\mathcal{C}$ is clear from context. There is a composition law for morphisms,

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

mapping $g \in \mathrm{Hom}(A, B)$ and $f \in \mathrm{Hom}(B, C)$, to $f \circ g \in \mathrm{Hom}(A, C)$, called the *composition of f and g*. Composition must be associative, and for every object A there must be an identity morphism, $\mathrm{Id}_A \in \mathrm{Hom}(A, A)$ so that

$$\mathrm{Id}_A \circ f = f \circ \mathrm{Id}_A = f.$$

Example A.1.1. The category of sets and set functions is a familiar category.

Example A.1.2. There are many familiar algebraic categories. The category of groups and group homomorphisms, the category of rings and ring homomorphisms, the category of fields and field homomorphisms and so on. Some of these categories have *subcategories*, for example the category of abelian groups and group homomorphisms is a subcategory of the category of groups and group homomorphisms. This is what is known as a *full subcategory*, since it consits of a subcollection of obejects, while the set of morphisms between any two objects remains the same. Similarly, the category of commutative rings and ring homomorphisms is a full subcategory of the category of rings and ring homomorphisms.

Example A.1.3. Any group G can be though of as a category with a single object $\{*\}$ and a morphism $g \in \mathrm{Hom}(*, *)$ for each element $g \in G$. Then the composition of two morphisms corresponds to multiplication in the group, and the identity morphism corresponds to the identity element. Of course, in order to form a group, the morphisms must also have inverse morphisms, a condition not neccesarily satisfied in a general category. The general category with exactly one element is the same thing as a semi-group with identity, also called a monoid.

Example A.1.4. The category of topological spaces and continuous maps will form a very important category in this text.

Example A.1.5. Any class of manifolds can form a category. There is the category of smooth manifolds and smooth maps, the category of topological manifolds and continuous maps, the category of C^r manifolds and C^r maps, the category of complex manifolds and holomorphic maps and so on.

There are many many more examples of categories. Often when speaking of a category, only the objects will be mentioned explicitly, and the morphisms will be assumed to be the appropriate structure preserving maps. For example, rather than saying the category of groups and group homomorphisms, one will simply refer to the category of groups, and the morphisms will be assumed to be the maps that preserve the group structure, i.e. the group homomorphisms.

A.2 Functors

The next most basic notion in category theory is that of a *functor* between two categories. Most of the invariants developed in algebraic topology are functors. A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$ and each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ in $\mathcal{C}$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ in $\mathcal{D}$, respecting composition and identities, i.e.

$$F(f \circ g) = F(f) \circ F(g)$$

and,

$$F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)} .$$

This is actually just one of two types of functors, called a *covariant functor*. The other type of functor is a *contravariant functor*. A contravariant functor associates each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ in $\mathcal{C}$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ in $\mathcal{D}$, and the composition laws change appropriately.

Example A.2.1. There is a *forgetful functor* from the category of topological spaces and continuous maps to the category of sets and set functions that sends each topological space to its underlying set and each continuous map to its underlying set function. It is called a forgetful functor since it "forgets" the extra structure of the topology. There are also forgetful functors from the categories of rings to groups, fields to rings, groups to sets, manifolds to topological spaces and so on.

Example A.2.2. Almost all of the topological invariants developed in this text are functors. Singular homology and cohomology are functors from the category of topological spaces to the category of graded abelian groups. Singular homology is a composition of the singular chain complex functor from topological spaces to the category of free abelian chain complexes, and the homology functor from the category of chain complexes to the category of graded abelian groups. Singular cohomology is a similar composition of functors. The other types of homology and cohomology are also functors. The fundamental group is also a functor. It is a functor from the category of topological spaces to the category of groups.

Example A.2.3. If we have two semi-groups with identity G_1 and G_2, thought of as categories as in example A.1.3, then any semi-group homomorphism $f : G_1 \to G_2$ gives a functor. It takes the single object of G_1 to the single object of G_2. Because it is a semi-group homomorphism, it takes the identity morphism in G_1 to the identity morphism in G_2. It takes compositions of morphisms to composition of morphisms, since composition of morphisms corresponds to multiplication in each of the respective semi-groups.

A.3 Monic, Epi and Isomorphism

In any category $\mathcal{C}$ a morphism $f : a \to b$ is called an *isomorphism* if there is another morphism $f' : b \to a$ in $\mathcal{C}$ such that $f' \circ f = \mathrm{Id}_a$ and $f \circ f' = \mathrm{Id}_b$. If such a morphism f' exists, it is unique and we write $f' = f^{-1}$. Two objects a and b are said to be *isomorphic* if there exists an isomorphism between them.

Definition A.3.1. A morphism $m : a \to b$ is said to be *monic* if it satisfies the following universal property: For any two morphisms $f_1, f_2 : c \to a$, $m \circ f_1 = m \circ f_2$ implies $f_1 = f_2$.

Definition A.3.2. A morphism $s : a \to b$ is said to be *epi* if it satisfies the following universal property: For any two morphisms $f_1, f_2 : b \to c$, $f_1 \circ s = f_2 \circ s$ implies $f_1 = f_2$.

Show that in the category ***Set***, with objects sets and morphisms set functions, a morphism is monic iff it is an injection and a morphism is epi iff it is a surjection. In ***Set*** a morphism is an isomorphim iff and it is monic and epi; however, this is not true in general. Can you think of an example?

B Direct Limits

A *partial order* on a set S is a binary relation $\leq$ on S with the following properties:

- $x \leq x$ for every $x \in S$.
- $x \leq y$ and $y \leq x$ implies $x = y$.
- $x \leq y$ and $y \leq z$ implies $x \leq z$.

A set S with a partial order $\leq$ is called a *partially ordered set* or *poset*. Given a set S, the subsets of S ordered by inclusion are an example of a poset. Notice that given two elements x and y in a poset, it is not neccesarily true that either $x \leq y$ or $y \leq x$, that is to say, there may be pairs of elements in a poset that are not comparable. A trivial example of a poset that illustrates this point is given by taking a set S with the relation $x \leq x$ for every $x \in S$ and no other relations. A *directed set* is a poset $\{S, \leq\}$ with the property that for any two elements $x, y \in S$ there is an element $z \in S$ such that $x \leq z$ and $y \leq z$. A trivial example of a directed set is given by taking any poset $\{S, \leq\}$ and adding one additional element, say M, with the property that $s \leq M$ for any $s \in S$. A subset $T \subset S$ of a directed set S is said to be *cofinal* in S if for every $s \in S$ there is an element $t \in T$ with $s \leq t$. In this case the restriction of the partial order to T makes T a directed set.

Let $(S, \leq)$ be a directed set. A *direct system of abelian groups* indexed by S is a family of abelian groups $\{A_s\}_{s \in S}$ along with group homomorphisms $f_{s,t} : A_s \to A_t$ for every pair $s, t \in S$ with $s \leq t$ such that for all $s \leq t \leq u$ we have $f_{s,u} = f_{t,u} \circ f_{s,t}$ and the map $f_{s,s}$ is the identity on A_s for any $s \in S$.

Given a directed set $(S, \leq)$ we can construct a category $\mathcal{D}_S$ with an object for each object of S and $\mathrm{Hom}(s,t) = \{r_{s,t}\}$ if $s \leq t$ and $\mathrm{Hom}(s,t) = \emptyset$ if $s \not\leq t$. Then a directed system of abelian groups indexed by S is a functor from $\mathcal{D}_S$ to the category of abelian groups and group homomorphisms.

Definition B.0.3. Let $\{A_s, f_{s,t}\}$ be a direct system of abelian groups indexed by a directed set S. The *direct limit* of this system $\varinjlim_S (A_s, f_{s,t})$ is an abelian group A together with homomorphisms $\rho_s : A_s \to A$ such that for all $s \leq t$ we have $\rho_t \circ f_{s,t} = \rho_s$ and A is universal with respect to this property. That is, if A' is an abelian group with homomorphisms $\phi_s : A_s \to A'$ satisfying $\phi_t \circ f_{s,t} = \rho_s$ for all $s \leq t$, then there is a unique homomorphism $\psi : A \to A'$ such that $\psi \circ \rho_s = \phi_s$ for all $s \in S$.

Proposition B.0.4. *Direct limits of abelian groups exist and are unique up to unique isomorphism.*

Proof. As usual with universal properties of this type, if the direct limit exists then it is unique up to unique isomorphism commuting with the structure maps ρ_s. To show that the direct limit exists consider $\tilde{A} = \oplus_{s \in S} A_s$ and introduce the equivalence relation generated by the following: For all $s \leq t$ and all $a_s \in A_s \subset \tilde{A}$, the element $a_s \in \tilde{A}$ is equivalent to $f_{s,t}(a_s) \in A_t \subset \tilde{A}$. Since the generators of this equivalence relation are additive, the quotient of $\tilde{A}$ by this relation is a quotient group. We claim that this quotient is the direct limit. To see this, first notice that we have the compositions $A_s \to \oplus_{s \in S} A_s = \tilde{A} \to \tilde{A}/\cong$, and these maps commute with the maps $f_{s,t}$. If we have maps $g_s : A_s \to B$ they define a map $\tilde{g} : \tilde{A} \to B$. If the g_s are compatible with the $f_{s,t}$ then $\tilde{g}$ factors uniquely through the quotient, showing that the quotient has the universal property. □

Example B.0.5. Consider the directed system of abelian groups indexed by $\{\mathbb{N}, \leq\}$ where all of the groups $A_i = \mathbb{Z}$ and the group homomorphisms are given by $f_{i,i+1} = \cdot(i+1)$, multiplication by $i+1$, and the neccesary compositions. Then $\varinjlim_{\mathbb{N}} \{\mathbb{Z}, f_{i,j}\} = \mathbb{Q}$.

Claim B.0.6. *Suppose that S is directed and $f \in S$ is a final element i.e. $s \leq f$ for every $s \in S$. Then for any directed system $\{A_s, \rho_{s,t}\}$ indexed by S, $\varinjlim_{\mathbb{S}} \{A_s, \rho_{s,t}\} = A_f$*

Exercise B.0.7. *Prove this claim.*

Let S be a directed set and $T \subset S$ a cofinal subset. Suppose that $\mathcal{A}_S = \{A_s, f_{s,t}\}$ is a direct system of abelian groups indexed by S. Then we define the restriction of $\mathcal{A}_S$ to T, to be the direct system $\mathcal{A}_T$ indexed by T obtained by considering only the abelian groups A_t for $t \in T$ and the homomorphisms $f_{t,t'}$ for $t, t' \in T$.

Lemma B.0.8. *If S is a directed set and $T \subset S$ a cofinal subset, $\mathcal{A}_S = \{A_s, f_{s,t}\}$ a direct system of abelian groups indexed by S, and $\mathcal{A}_T$ the restriction of $\mathcal{A}_S$ to T, then the direct limit of $\mathcal{A}_S$ and $\mathcal{A}_T$ are canonically identified.*

Proof. For each $t \in T$ we have the composition $A_t \to \tilde{A} \to \varinjlim_S \{A_w, f_{s,s'}\}$. These are compatible with the $f_{t,t'}$ for $t \leq t'$ elements of T, and hence they determine a map $p : \varinjlim_T \{a_t, f_{t,t'}\} \to \varinjlim_S \{A_s, f_{s,s'}\}$. Given $a \in A_s$ there is $t \in T$ with $s \leq t$ and hence in $\varinjlim_S \{A_s, f_{s,s'}\}$ the class represented by a is also represented by $f_{s,t}(a) \in A_t$. This implies that p is surjective. If $b \in A_t$ represents the trivial element in $\varinjlim_S \{A_s, f_{s,s'}\}$ then for some $s \in S$ with $t \leq s$ we have $f_{t,s}(b) = 0$. But there is $t' \in T$ with $s \leq t'$. Clearly, $f_{t,t'}(b) = f(s, t'(f_{t,s}(b) = 0$, so that b also represents 0 in $\varinjlim_T \{A_t, f_{t,t'}\}$. This proves that p is one-to-one, and consequently that it is an isomorphism. ☐

Remark B.0.9. One can define direct systems in any abelian category. For categories such as modules over a ring, or vector spaces over a field, direct limits exist and are defined by the same construction as given above.

There is another type of direct limit that will be important for us: direct limits in the category of topological spaces. Let S be a directed set and $\{X_s, f_{s,s'}\}$ a directed system of topological spaces and continuous maps indexed by S. This means that for each $s \in S$, X_s is a topological space and for each $s \leq s'$ we have a continuous map $f_{s,s'} : X_s \to X_{s'}$ such that $f_{s,s} = \mathrm{Id}_{X_s}$ and $f_{s',s''} \circ f_{s,s} = f_{s,s''}$. Then the direct limit $\varinjlim_S \{X_s, f_{s,s'}\}$ is defined to be the quotient space of

$$\coprod_{s \in S} X_s$$

by the equivalence relation generated by the following: for any $s \leq s'$ and any $x \in X_s$, the point x is equivalent to $f_{s,s'}(x) \in X_{s'}$.

If the $f_{s,s'}$ are embeddings, then each X_s embeds in the direct limit and the direct limit X is the union of these embeddings. It has the weak topology induced from these subspaces: that is to say a subset of X is open if and only if its intersection with each $X_s \subset X$ is open in X_s.

B.1 Direct Limits and singular homology

An important property of homolgy is that it behaves well under the taking of direct limits. Let X be a Hausdorff space that is an increasing union of closed subspaces $\{X_n\}_{\{n=1,2,\cdots\}}$. We suppose that X has the direct limit or weak topology which means that a subset $U \subset X$ is open if and only if $U \cap X_n$ is open for all $n \geq 1$.

Lemma B.1.1. *Then*

$$H_*(X) = \varinjlim_n H_*(X_n),$$

where the maps in the direct system of homology groups are the maps induced by the inclusion.

Proof. The inclusions $X_n \subset X$ are continuous and hence induce maps $j_n \colon H_*(X_n) \to H_*(X)$. Clearly, these are compatible with the inclusions of $X_n \subset X_m$. Hence, they define a map

$$\varinjlim_n H_*(X_n) \to H_*(X).$$

We will show that this map is an isomorphism.

Lemma B.1.2. *Let $K \subset X$ be compact. Then $K \subset X_n$ for some n.*

Proof. Let $K \subset X$ be a compact subset and suppose that $K \not\subset X_n$ for any n. Then there is a sequence $\{a_n\}_{n=1}^{\infty}$ of points in K with $a_n \notin X_n$. We claim that $V_N = X \setminus \cup_{n=N}^{\infty}\{x_n\}$ is an open subset of X. Of course, $V_N \cap X_t$ is the complement in X_t of a finite set. Since the X_t are Hausdorff, it follows that $V_N \cap X_t$ is open in X_t for all N and t. Because the topology on X is the weak topology, it follows that $V_N \subset X$ is open for all N. Clearly, $V_N \subset V_{N+1}$ and $\cup_N V_N = X$. Thus, $\{V_N \cap K\}$ form an increasing open covering of K. By the compactness of K, it follows that this cover has a finite subcover, which, because the V_N are an increasing sequence of subsets, implies that $K \subset V_N$ for some N. This is absurd, since $x_n \in K \setminus V_N$ for all $n > N$.

□

Since any singular chain in X has compact support this implies that any singular chain in X is contained in X_n for some n. Thus, any $\alpha \in H_*(X)$ has a representative cycle ζ in X_n for some n, and hence is in the image of $H_*(X_n) \to H_*(X)$, and therefore in the image of $\varinjlim_n H_*(X_n) \to H_*(X)$. On the other hand if ζ is a cycle in X_n which is homologous to zero in X, let c be a chain in X with $\partial c = \zeta$. Then c lies in X_m for some $m > n$ and hence $[\zeta] = 0$ in $H_*(X_m)$ and so $[\zeta] = 0$ in the direct limit. This proves that the map $\varinjlim_n H_*(X_n) \to H_*(X)$ is one-to-one. □

C CW Complexes

A *CW complex* X is a topological space made from the union of an increasing sequence of closed subspaces

$$X^{-1} = \emptyset \subset X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset \cdots$$

X^0 consists of a set of points, and X^n is obtained from X^{n-1} by attaching a collection of n-cells, $\{D^n_\alpha\}_{\alpha \in A}$, via attaching maps $\{\phi_\alpha : \partial D^n_\alpha \to X^{n-1}\}$. By this mean we mean that

$$X^n = \frac{\coprod_{\{\alpha \in A\}} D^n_\alpha}{x \in \partial D_\alpha \backsim \phi_\alpha(x) \in X^{n-1}}$$

X is a *finite CW complex* if X=X^n for some n, and X^k is obtained from X^{k-1} by attaching a finite number of k-cells for each k. For a finite CW complex the topology is given by first taking the disjoint union topology, and then the quotient topology, when attaching cells. An inifinite CW-complex is given the weak topology, i.e. a set $U \subset X$ is open if and only if $U \cap X^n$ is open in X^n for each n. The subspace X^n of X is called the *n-skeleton* of X.

D Simplicial Complexes

D.1 The Definition

A simplicial complex K consists of a set $V = V(K)$, whose elements are called the *vertices* of K and a set $\mathcal{S}(K) \subset 2^V$ of subsets of V subject to the following requirements:

- If $s \in \mathcal{S}$ then the cardinality of s is finite.
- $\emptyset \notin \mathcal{S}$.
- If $s \in \mathcal{S}$ and if $t \subset s$ is a non-empty subset of s then $t \in \mathcal{S}$.
- Every subset of 2^V of cardinality 1 is a member of $\mathcal{S}$.

The elements of $\mathcal{S}$ are called the simplices of K. The *dimension* of a simplex is one less than its cardinality as a subset of V. We implicitly identity the simplices of K of dimension zero with the corresponding vertex of K.

A subcomplex of K is a subset $V' \subset V(K)$ and a subset $\mathcal{S}' \subset 2^{V'}$ which makes a simplicial complex and such that every $s \in \mathcal{S}'$ is also an element of $\mathcal{S}(K)$. A simplex t is said to be a *face* of another simplex s, denoted $t < s$, if t is a subset of s. A *facet* is a codimension-one face.

A *finite simplicial complex* is one whose vertex set is finite.

A *simplicial map* $\phi : K \to L$ between simplicial complexes is a set function $V(\phi) : V(K) \to V(L)$ with the property that for every $s \in \mathcal{S}(K)$ the image $\phi(s)$ is an element of $\mathcal{S}(L)$. (We do not require that $\phi(s)$ have the same cardinality as s.)

These objects and maps form a category: one has the obvious associative operation of composing simplicial maps.

D.2 The Geometric Realization

For a finite set V, the simplex $\Delta(V)$ spanned by V is the finite subcomplex with V as its set of vertices and $\mathcal{S}(\Delta(V)) = 2^V \setminus \emptyset$ as its set of simplices. Notice that any finite simplicial complex K with vertex set V is identified with a subcomplex of $\Delta(V)$.

Every simplicial complex K determines a topological space $|K|$, called its *geometric realization*. Given a simplex $s \in S(K)$, with vertices $V(s)$ we define the geometric realization of s to be the subset of the real vector space $\mathbb{R}^s$ with basis $V(s)$, given by $|s| = \left\{ \sum_{v \in V(s)} t_v v \middle| t_v \geq 0 \ \forall v \in V(s), \ \sum_{v \in V(s)} t_v = 1 \right\}$. In the special case that $V(s) = \{0, \cdots, n\}$, the geometric realization of s is just the standard n-simplex, Δ^n. indexsimplicial complex!geometric realization Now, to define $|K|$, start with the topological space

$$X(K) = \coprod_{s \in \mathcal{S}} |s|,$$

where the topology is the disjoint union of the standard subspace topologies on $|s|$ from the embeddings $|s| \subset \mathbb{R}^s$. Then introduce an equivalence relation on X generated by the following relation: if $t < s$ then $|t|$ is identified with the geometric face of $|s|$ spanned by the vertices of t. We denote by $|K|$ the quotient space with the quotient topology. A point of $|K|$ can be written uniquely as $\sum_{v \in V} \lambda_v v$ where the λ_v are non-negative real numbers which sum to 1 and such that the set of $v \in V$ for which $\lambda_v \neq 0$ are the vertices of a simplex of K. The set of such sums $\{\sum_{v \in V} \lambda_v v | \{v | \lambda_v \neq 0\} \subset s\}$ is exactly the image of $|s| \subset |K|$. In fact, this gives an embedding of $|s| \to |K|$ whose image is a closed subset. This image is called the closed simplex $|s|$. The open simplex $\text{int}|s|$ is the closed simplex $|s|$ minus the union of all the closed simplices associated with the proper faces of s. Then $\text{int}|s| = \{\sum_{v \in V} \lambda_v v | \lambda_v \neq 0 \text{ iff } v \in s\}$.

The geometric realization $|K|$ is the union of its closed simplices and has the induced topology: a subset $U \subset |K|$ is open if and only if its intersection with each closed simplex is an open subset of that closed simplex. It is also true that $|K|$ is the union of its open simplices and that each point is contained in exactly one open simplex.

The finite subcomplexes of K form a directed set under inclusion, and as the next lemma shows, K is their direct limit.

Lemma D.2.1. *$|K|$ is the union of $|K_f|$ as K_f runs over the finite subcomplexes of K, and the topology of $|K|$ is the weak topology induced from this union; i.e., a subset $U \subset |K|$ is open if and only if its intersection with each $|K_f|$ is an open subset of $|K_f|$*

Exercise D.2.2. *Prove this lemma.*

A simplicial map $\phi : K \to L$ clearly induces a continuous map $|\phi| : |K| \to |L|$ which sends each closed simplex $|s|$ affine linearly onto the closed simplex $|\phi(s)|$ of $|L|$. The correspondences define a functor from the category of simplicial complexes and simplicial maps to the category of topological spaces and continuous maps.

Exercise D.2.3. *Show that for any simplicial complex K, the topological space $|K|$ is Hausdorff.*

Exercise D.2.4. *Show that if L is a subcomplex of K, then $|L|$ is a closed subset of $|K|$.*

Exercise D.2.5. *Show that for a finite simplicial complex K, $|K|$ is a compact metric space if and only if $|K|$ is a finite complex.*

Exercise D.2.6. *Show that for any simplex s of K, the space $|s|$ is a closed subspace of $|K|$*

Exercise D.2.7. *Show that for simplices s, t of K the intersection $|s| \cap |t|$ is equal to $|s \cap t|$. In particular, the intersection of two closed simplices is a (closed) face of each.*

Exercise D.2.8. *Show that direct limits exist in the category of simplicial complexes and simplicial maps and that*

$$|\lim_S(K_s, \phi_{s,s'})| = \varinjlim_S(|K_s|, |\phi_{s,s'}|).$$

D.3 Subdivision

Let K be a simplicial complex and let $|K|$ be its geometric realization. A subdivision K' of K is another simplicial complex whose vertices are points of $|K|$ with the property that if σ' is a simplex of K' then there is a closed simplex σ of $|K|$ that contains all the vertices of σ'. Using this we can define a continuous mapping $|K'| \to |K|$ compatible with the natural map on the vertices sending each closded simplex $|\sigma'|$ of $|K'|$ linearly onto a convex subset of a closed simplex $|\sigma|$ of $|K|$. The last condition for K' to be a subdivision of K is that this map be a homeomorphism. Notice that if K' is a subdivision of K then we have an identification $|K'| = |K|$. This identification is linear on each simplex of $|K'|$.

E Smooth Manifolds and Smooth Maps

Our motivation for defining smooth manifolds is to try and capture the class of topological spaces on which it is possible to apply the tools of calculus that have been developed for use in the Euclidean spaces, $\mathbb{R}^n$. Before we can define this class of spaces, we need a few preliminary defintions.

Definition E.0.1. A paracompact Hausdorff space M is called a *topological manifold of dimension n* if every point $p \in M$ has an open neighborhood, that is homeomorphic to an open subset of $\mathbb{R}^n$.

Although topological manifolds have many nice properties, this is far too general a class of spaces for us to hope to be able to extend the tools of calculus to. We require some additional structure, namely:

Definition E.0.2. A *smooth structure* $\mathcal{F}$ on an n-dimensional topological manifold M is a family of pairs (U_i, ϕ_i) such that:

1. $\{U_i\}$ forms an open cover of M.
2. $\phi_i : U_i \to V_i$ is a homeomorphism from U_i to an open subset $V_i \subset \mathbb{R}^n \ \forall\ i$.
3. $\psi_{ij} = \phi_i \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is infinitely differentiable.
4. The family $\mathcal{F}$ is maximal with respect to (3) i.e if we have any pair (U, ϕ) where $U \subset M$ is open and $\phi : U \to V$ is a homeomorphism from U to an open subset in $\mathbb{R}^n$, and $\phi \circ \phi_i^{-1}$ and $\phi_i \circ \phi^{-1}$ are both $C^\infty\ \forall\ \phi_i \in \mathcal{F}$, then $(U, \phi) \in \mathcal{F}$

A collection of pairs $\{(U_i, \phi_i)\}$ satisfying properties (1) through (3) is called an *atlas*, and so a differentiable structure is sometimes also called a *maximal atlas*. A pair (U_i, ϕ_i) is called a *chart*. The subset U_i is called a *coordinate neighborhood* or *coordinate patch*. The map ϕ_i is called a *coordinate function* or just a *coordinate*. We can write ϕ_i in terms of its components as $\phi_i(p) = \{x_i^1(p), x_i^2(p), \dots, x_i^n(p)\}$. Then the $\{x_i^j\}_{j=1}^n$ are also called *coordinates*. The functions ψ_{ij} are called *transition functions* or *gluing functions*. The following lemma shows that any atlas gives rise to a unique smooth structure.

Lemma E.0.3. *Given any atlas $\mathcal{F}_0 = \{(U_i, \phi_i)\}$ there is a unique differentiable structure $\mathcal{F}$ containing $\mathcal{F}_0$.*

Proof. Let

$$\mathcal{F} = \{\text{All charts } (U, \phi) \text{ on } M \mid \phi \circ \phi_i^{-1} \text{ and } \phi_i \circ \phi^{-1} \text{ are both } C^\infty\ \forall\ \phi_i \in \mathcal{F}_0\}$$

The proof of uniqueness of this structure is left as an exercise. □

Two atlases, $\mathcal{F} = \{(U_i, \phi_i)\}$ and $\mathcal{F}' = \{(V_j, \psi_j)\}$, on a manifold M are said to be *compatible* if $\psi_j \phi_i^{-1} : \phi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j)$ and $\phi_i \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \phi_i(U_i \cap V_j)$ are infinitely differentiable whenever $U_i \cap V_j \neq \emptyset$. It is easy to see that two compatible atlases give rise to the same differentiable structure on M. We now have all of the neccesary definitions to define the class of smooth manifolds.

Definition E.0.4. A *smooth n-manifold* is a pair $(M, \mathcal{F})$ where M is an n-dimensional topological manifold, and $\mathcal{F}$ is a differentiable structure on M.

Frequently, the manifold is simply referred to as M when the differentiable structure is understood; however, be aware that a given space can have many possible differentiable structures. For example, Milnor showed in 1956 that S^7 has 28 differentiable structures, and it was later discovered that $\mathbb{R}^4$ has infinitely many differentiable structures.

Now that we have defined a class of objects, we would like to specify a collection of morphisms between them so that we can work in the category of smooth manifolds and smooth maps. Let $f : M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. Let $p \in M$, and (U, ϕ) and (V, ψ) be charts on M and N respectively such that $p \in U$ and $f(p) \in V$. Then f has a coordinate presentation at p as $\psi f \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$. If we write $\phi(p) = \{x^\mu\}$ and $\psi(f(p)) = \{y^\nu\}$, then $\psi f \phi^{-1}$ is the vector-valued function $y = \psi f \phi^{-1}(x)$. We will frequently abuse notation and simply write $y = f(x)$ or $y^\nu = f^\nu(x^\mu)$ when the coordinate sytsems are understood.

Definition E.0.5. A map $f : M \to N$ between smooth manifolds is said to be *smooth at* p, for some point $p \in M$, if $\psi f \phi^{-1}$ is C^∞ for some coordinate presentation of f at p. We say f is *smooth* if f is smooth at all points $p \in M$.

Claim E.0.6. *The smoothness of f is independent of the coordinates in which it is presented.*

Proof. Suppose that we have two overlapping charts, (U_1, ϕ_1) and (U_2, ϕ_2) with a point $p \in U_1 \cap U_2$. Suppose that f is smooth with respect to ϕ_1 i.e. $\psi f \phi_1^{-1}$ is C^∞. Then we have $\psi f \phi_2^{-1} = \psi f \phi_1^{-1}(\phi_1 \phi_2^{-1})$ is also C^∞ since the transition function $\psi_{12} = \phi_1 \phi_2^{-1}$ is smooth by defintion. Thus, f is also smooth with respect to ϕ_2. The same idea can be used to show that the smoothness of f also does not depend on the chart in N. □

It is an easy exercise to show that the composition of two smooth maps is again smooth, and the identity map on any smooth manifold is smooth. Associativity follows from the associativity of the underlying set functions. Thus, the collection of smooth manifolds and smooth maps of manifolds defines a category.

Definition E.0.7. A map $f : X \to Y$ that is a smooth homeomorphism and has a smooth inverse is a *diffeomorphism*.

Remark E.0.8. Diffeomorphisms are the isomorphisms in the category of smooth manifolds. A smooth homeomorphism is not neccesarily a diffeomorphism, as the following example illustrates.

Example E.0.9. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $t \mapsto t^3$. This map is clearly a smooth homemorphism from $\mathbb{R}$ to itself; however, $f^{-1}(t) = \sqrt[3]{t}$ is not smooth since it is not differentiable at $t = 0$. Thus, f is not a diffeomorphism.

F Germs and Sheaves

The first major tool of real calculus is the derivative. This is a linear map associated to each continuous function. In order to apply this tool in the context of smooth manifolds we will need to associate a real vector space to each point on our manifold, called the tangent space, and to each smooth map between manifolds, a linear map between the tangent spaces of those manifolds. Our definition of the tangent space to a manifold will depend only on the local properties of functions on the manifold. To formalize this we will introduce the notion of a *germ of a continuous function*. This leads us naturally to the notion of a sheaf on a topological space. Without moving too far into the language of sheaves, we will make use of one specific example, the sheaf of local continuous functions on a topological space. We will then specify to the sheaf of smooth functions on a smooth manifold. These will also allow us to formulate a second defintion of smooth manifold.

F.1 The sheaf of local continuous functions

Let X be a topological space. We begin with a category $\mathcal{T}(X)$ whose objects are the open subsets of X and whose morphisms are inclusions. That is to say, given U and V in X open, if $U \subset V$ then there is a single morphism from U to V, thought of as inclusion, and otherwise $Hom(U,V) = \emptyset$. We also have the category $\mathcal{A}$ of $\mathbb{R}$-algebras and $\mathbb{R}$-algebra homomorphisms. For nice spaces, for example in the case X is a compact Hausdorff space, the algebra of continuous functions on that space, $\mathcal{C}(X)$ contains all of the information about that space, namely, the points of X and the toplogy on X. We express this formally in the folowing two propositions.

Proposition F.1.1. *Let X be a compact Hausdorff space and $\mathcal{C}(X)$ be the algebra of continuous functions on X. Then the points of X correspond to the maximal ideals of $\mathcal{C}(X)$.*

Proof. Let m be a maximal ideal of $\mathcal{C}(X)$. If $f \in m$, then $\exists\, x \in X$ such that $f(x) = 0$. To see this, suppose $f \in m$, but f is nowhere zero. Then there is a function $g \in \mathcal{C}(X)$ given by $g(x) = (f(x))^{-1}$. Then,

$$g(x)f(x) = (f(x))^{-1}f(x) = 1 \in m.$$

This contradicts the maximality of m.

Now we claim that there exists a point $x_0 \in X$ such that $f(x_0) = 0 \;\forall\; f \in m$. Suppose not. Then for any point $x \in X$, there is a function $f_x \in m$ so that $f_x(x) \neq 0$. Since the functions are continuous, $\forall\; x \in X$ there is an open neighborhood of x, U_x, such that $f_x(u) \neq 0 \;\forall\; u \in U_x$. The collection $\{U_x\}_{x \in X}$ cover X, and since X is compact, we can take a finite subcover, $\{U_i\}_{1 \le i \le N}$ where $f_i \neq 0$ on U_i. Then consider the function f given by $f(p) = \sum_{i=1}^{N} (f_i(p))^2$. Since for each i, $(f_i(p))^2 \geq 0$ on all of X, and for every point in X, at least one of the f_i is strictly greater than 0, we see that $f > 0$ on all of X. But $f \in m$, so this is a contradiction. So for any maximal ideal m we have shown that there is at least one point of X at which all the functions in m vanish. Suppose that m is a maximal ideal in $\mathcal{C}(X)$, and there are two distinct points, $x, y \in X$ such that every $f(x) = f(y) = 0$ for every $f \in m$. Since X is a compact Hausdorff space, X is metrizable, and thus the function d_x given by the distance from x is a continuous function which is zero at x, but non-zero at y. Then clearly the ideal consisting of all functions that vanish at x strictly contains m, contradicting the maximality of m. So in fact, for any maximal ideal m we have shown that there is exactly one point of X at which all the functions in m vanish.

Given a point $x \in X$, the set $m_x = \{f \in \mathcal{C}(X) | f(x) = 0\}$ is a maximal ideal of $\mathcal{C}(X)$, since $m_x = ker(\phi)$, where $\phi : \mathcal{C}(X) \to \mathbb{R}$ is the surjective ring homomorphism given by $\phi(f) = f(x)$, and thus $\mathcal{C}(X)/m_x \cong \mathbb{R}$. Since $\mathbb{R}$ is a field, this implies m_x is a maximal ideal. $\square$

Proposition F.1.2. *Given $\mathcal{C}(X)$, we can recover the topology on X.*

Proof. Given a function $f \in \mathcal{C}(X)$, the set, $\{x \in X | f(x) = 0\}$ is a closed set of X. Given a closed set of $C \subset X$, we can define a continuous function that is zero on exactly the points of C (Why?). $\square$

We have a contravariant functor $\mathcal{S} : \mathcal{T}(X) \to \mathcal{A}$ given by $U \mapsto \mathcal{C}(U) =$ the algebra of continuous functions on U, and to an inclusion $U \hookrightarrow V$ associates the restriction mapping $r_{V,U}$ from functions on V to functions on U. To say this another way, to each open subset U of X we have the algebra $\mathcal{C}(U)$ of continuous functions on U and for each inclusion $U \subset V$ we have the restriction mapping $r_{V,U} : \mathcal{C}(V) \to \mathcal{C}(U)$ with the property that $r_{U,W} \circ r_{V,U} = r_{V,W}$. The functor $\mathcal{S}$ is called the *sheaf of local continuous functions*. Now let $x \in X$. We have a directed system of open neighborhoods of x defined as follows: We say that $V \geq U$ if $U \subset V$. Clearly, this relation is transitive and the system is directed in the sense that if U_1 and U_2 are open neighborhoods of x, then there is an open neighborhood V of x with $V \geq U_1$ and $V \geq U_2$. We define the *germ of a continuous function at* x as the direct limit over the directed system of open neighborhoods of x of the continuous functions on those neighborhoods. A germ is represented by an open neighborhood of U of x and a continuous real-valued function f on U. Two representatives (U, f) and (V, g) represent the same germ if and only if there is an open neighborhood $W$of x contained in $U \cap V$ with $f|_W = g|_W$.

Exercise F.1.3. *Show that the germ at x has a value at x, namely the value of any representative of that germ at x, but it does not have a well-defined value at any other point.*

Suppose that $f : X \to Y$ is a continuous mapping between two topological spaces. Then by pullback it induces a map from the algebra of continuous functions on an open subset $U \subseteq Y$ to the algebra of continuous functions on the open subset $f^{-1}(U) \subseteq X$. This association is compatible with restrictions and hence can be thought of as a map from the algebra of germs of continuous functions on Y at $y = f(x)$ to the algebra of germs of continuous functions on X at x. If f is a homeomorphism, then pullback by f induces an isomorphism from the sheaf of continuous functions on Y to the sheaf of continuous functions on X.

F.2 The sheaf of local $\mathcal{C}^\infty$ functions on a smooth manifold

Let M be a smooth manifold. For each open subset $U \subseteq M$ we have the ring of smooth $(\mathcal{C}^\infty)$ functions on U. These are closed under the restriction mappings $r_{U,V}$ used in the definition of the sheaf of continuous functons on M. Hence, we have a functor from the category of open subsets of M to the category of $\mathbb{R}-$algebras, which we call the sheaf of smooth (or $\mathcal{C}^\infty$) functions on M. In a natural way it is a subsheaf of the sheaf of continuous functions on M.

We can turn this process around. Suppose that we have a topological space X and a subsheaf $\mathcal{S}'$ of the sheaf $\mathcal{S}$ of continuous functions on X. That is to say, for each open set $U \subseteq X$ we have a subalgebra of the algebra of continuous functions on U, with the property that these subalgebras are closed under the restrictions $r_{V,U}$. Suppose further that locally this subsheaf is isomorphic to the sheaf of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$. By this we mean that for each $x \in X$ there is an open neighborhood $U \subset X$ and a homeomorphism from U to an open subset $V \subset \mathbb{R}^n$ which induces an isomorphism between the restriction of the sheaf $\mathcal{S}'$ to U and the sheaf of $\mathcal{C}^\infty$-functions on V. Then we can use these homeomorphisms to define local coordinates on X. In particular, the local coordinate functions of these charts are elements of the sheaf $\mathcal{S}'$. It then follows that on the intersection of two of these coordinate patches, the coordinate functions from one patch to another are $\mathcal{C}^\infty$ with respect to the local coordinates of the other patch. Hence, the transition functions are $\mathcal{C}^\infty$ and so we have determined a smooth structure on X. Thus, if X is paracompact and Hausdorff, we will have determined a smooth structure of a smooth manifold on X by specifying a subsheaf of the sheaf of continuous functions on X, the sheaf of functions that are to be smooth in the structure that we are constructing.

F.3 The tangent space

We will now use the language developed in the previous sections to define the tangent space. Let M be a smooth manifold and $x \in M$ be a point on M. We have defined the germ of a continuous function at x. Now consider the collection of all germs of smooth functions at x. This set of germs inherits the structure of an $\mathbb{R}$-algebra from the $\mathbb{R}$-algebra structure on representatives.

Definition F.3.1. A *local derivation* at $x \in M$ is an $\mathbb{R}$-linear map $\mathcal{D} : \{\text{germs at } x\} \to \mathbb{R}$ satisfying the Leibniz rule:

$$\mathcal{D}(f \cdot g)(x) = f(x) \cdot \mathcal{D}(g)(x) + g(x) \cdot \mathcal{D}(f)(x)$$

Exercise F.3.2. *Show that the local derivations at $x \in M$ form a $\mathbb{R}$-vector space*

Exercise F.3.3. *Suppose that M is an open subset of $\mathbb{R}^n$ and $x \in M$. Show that $\frac{\partial}{\partial x_i}|_x$ (i.e. the operation that assigns to a differentiable function f defined near x the number $\frac{\partial f}{\partial x_i}(x) \in \mathbb{R}$) is a local derivation.*

Exercise F.3.4. *Suppose that M and N are smooth n-manifolds and $x \in M$, $y \in N$ are points with open neighborhoods $U \subset M$ of x and $V \subset N$ of y. Suppose also that there exists a diffeomorphism $\phi : U \to V$ so that $\phi(x) = y$. Show that ϕ induces an isomorphism from the algebra of germs at y to the algebra of germs at x, and hence a linear isomorphism from the vector space of local derivations at x to the vector space of local derivations at y.*

Now we have a fundamental lemma:

Lemma F.3.5. *Suppose that $U \subset \mathbb{R}^n$ is an open neighborhood of 0 and $g : U \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function with $g(0) = 0$. Then there exists a neighborhood $V \subset U$ of 0 and $\mathcal{C}^\infty$ functions $h_i : V \to \mathbb{R}$ such that $g|_V = \sum x_i h_i(x)$.*

Exercise F.3.6. *Prove this lemma.*
Hint: Let $V \subset U$ be an open neighborhood centered at 0. Then by the fundamental theorem of calculus for an $x \neq 0$

$$g(x) = \int_0^1 \frac{\partial g}{\partial \overrightarrow{x}}(tx)dt$$

where $\frac{\partial g}{\partial \overrightarrow{x}}$ is the directional derivative. That is, if $x = (x^1, \ldots, x^n)$, then

$$\frac{\partial g}{\partial \overrightarrow{x}} = \sum x_i \frac{\partial g}{\partial e_i},$$

where e_i is the i-th standard basis element of $\mathbb{R}^n$. Define

$$h_i(x) = \int_0^1 \frac{\partial g}{\partial e_i}(tx)dt$$

Show h_i is $\mathcal{C}^\infty$ and $\sum x_i h_i(x) = g(x)$.

Exercise F.3.7. *Now show that if in addition $\frac{\partial g}{\partial x_i}(0) = 0$ then there exist $\mathcal{C}^\infty$ functions $h_{ij}(x)$ for $1 \leq i, j \leq n$ with $g(x) = \sum_{i,j} x^i x^j h_{ij}(x)$.*

Exercise F.3.8. *Use the previous exercise to show that $\frac{\partial}{\partial x^i}|_0$ form a basis for the local derivations of $\mathbb{R}^n$ at 0.*

Exercise F.3.9. *Show that if M is an n-manifold then for any $x \in M$ the vector space of local derivations at x is an n-dimensional real vector space. Show that if $(x^i, \ldots, x^n)$ are local $\mathcal{C}^\infty$-coordinates defined on some neighborhood of x then $\{\frac{\partial}{\partial x^1}|_x, \ldots, \frac{\partial}{\partial x^n}|_x\}$ are a basis for this $\mathbb{R}$-vector space.*

Definition F.3.10. The n-dimensional real vector space of local derivations at a point $x \in M$ is called the *tangent space to M at x* and is denoted TM_x.

As an example, suppose that we have an n-dimensional manifold sitting inside some higher dimensional Euclidean space, $M^n \subset \mathbb{R}^N$. Then TM_x is an n-dimensional linear supspace of $\mathbb{R}^N$ through the origin. Think of the n-dimensional hyperplane tangent to M at x, translated to the origin.

Let M and N be smooth manifolds of dimensions m and n respectively. Let $f : M \to N$ be a smooth map. Suppose $x \in M$ and $y = f(x) \in N$.

Exercise F.3.11. *Show that pre-composition with f defines an $\mathbb{R}$-algebra homomorphism from germs of $\mathcal{C}^\infty$-functions on N at y to germs of $\mathcal{C}^\infty$-functions on M at x*

Exercise F.3.12. *Show f induces an $\mathbb{R}$-linear map, denoted $Df_x : TM_x \to TN_y$.*

Definition F.3.13. The $\mathbb{R}$-linear map, $Df_x : TM_x \to TN_y$ induced by f is called the *differential of f at x.*

Exercise F.3.14. *Let P be another smooth manifold. Show that if $g : N \to P$ is smooth with $g(y) = z$ then $Dg_y \circ Df_x = D(f \circ g)_x : TM_x \to TP_z$.*

F.4 Variation of the tangent space with $x \in M$

Let $U \subset \mathbb{R}^n$ be an open set. Then for every point $x \in U$ we have a basis $\{\frac{\partial}{\partial x^1}|_x, \ldots, \frac{\partial}{\partial x^n}|_x\}$ for TU_x. We view this basis as giving an isomorphism $\phi_x : \mathbb{R}^n \to TU_x$ defined by $\phi_x(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i \frac{\partial}{\partial x_i}|_x$

Thus, as x varies we define $\phi : U \times \mathbb{R}^n \to \underset{x \in U}{\cup} TU_x$ by

$$\phi(u, \overrightarrow{t}) = \phi_u(\overrightarrow{t}) = \sum_{i=1}^{n} t_i \frac{\partial}{\partial x^i}|_u$$

The map ϕ induces a topology and even a smooth structure on $\underset{x \in U}{\cup} TU_x$. Now suppose $U, V \subset \mathbb{R}^n$ are open sets and $\psi : U \to V$ is a diffeomorphism. Then we have $\underset{x \in U}{\cup} D\psi_x : \underset{x \in U}{\cup} TU_x \to \underset{y \in V}{\cup} TV_y$ which sends $\tau \in TU_x$ to $D\psi_x(\tau) \in TU_y$. By construction it is a family of linear isomorphisms. We wish to consider the composition:

$$\begin{array}{ccc}
\underset{x \in U}{\cup} TU_x & \xrightarrow{D\psi} & \underset{y \in V}{\cup} TV_y \\
\cong\uparrow & & \cong\uparrow \\
U \times \mathbb{R}^n & \xrightarrow{D\psi} & V \times \mathbb{R}^n
\end{array}$$

Claim F.4.1. *This composition sends*

$$(u, \overrightarrow{t}) \mapsto (\psi(u), \sum_j \frac{\partial \psi}{\partial x^j}(u) t^j)$$

Exercise F.4.2. *Use the chain rule to prove this claim.*

As a consequence of this formula we see that $D\psi$ is a diffeomorphism. We have a commutative diagram of smooth maps:

$$\begin{array}{ccc}
\bigcup_{x\in U} TU_x & \longrightarrow & \bigcup_{y\in V} TV_y \\
\big\uparrow & & \big\downarrow \\
U \times \mathbb{R}^n & \xrightarrow{D\psi} & V \times \mathbb{R}^n \\
{\scriptstyle P_1}\big\downarrow & & {\scriptstyle P_1'}\big\downarrow \\
U & \xrightarrow{\psi} & V
\end{array}$$

G Vector Bundles

A real vector bundle is a family of vector spaces parameterized by some base space with some additional structure.

Definition G.0.3. A *real vector bundle* consists of two topological spaces, E and B, and a continuous surjective map $\pi : E \to B$ with the following properties:

1. $\pi^{-1}(b) \subset E$ is a real vector space for each point $b \in B$.
2. Addition and scalar multiplication in E are continuous. That is, the maps

$$\mathbb{R} \times E \to E$$
$$(\lambda, e) \mapsto \lambda \cdot e$$

   and

$$E \times_B E \to E$$
$$(e_1, e_2) \mapsto e_1 + e_2$$

   are continuous.
3. There exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of B so that $\pi^{-1}(U_\alpha) \cong U_\alpha \times V$ for some real vector space V.

The space E is called the *total space*. The space B is called the *base space*. The map π is called the *projection map*. The vector spaces $\pi^{-1}(b)$ are called sl fibers. Condition 3 in the definition is called *local triviality*.

Example G.0.4. Given any topological space B one can form the trivial bundle over B by taking the total space to be $B \times V$ for some real vector space V, and the projection map $\pi : B \times V \to B$ to be just projection onto the first coordinate.

An alternative way to present a vector bundle is to start with the base space B, an open cover $\{U_\alpha\}_{\alpha\in A}$ of B and a vector space V. Take trivial vector bundles over each of the open sets U_α, and then glue them togther. To do this for any pair of open sets U_α

For any functorial operation on vector spaces there exists a corresponding

H Integration of Differential Forms

Let $U \subset \mathbb{R}^n$ be an open subset and let $\omega \in \Omega^n(U)$ be a differential n-form on U. Then there is a C^∞-function $f : U \to \mathbb{R}$ such that

$$\omega(x) = f(x) dx_1 \wedge \cdots \wedge dx_n.$$

If ω has compact supports in U then we define

$$\int_U \omega = \int_U f(x) dx_1 \cdots dx_n$$

where the integral on the right-hand-side is the usual Lebesgue (or riemann) integral in Euclidean space.

Lemma H.0.5. *Let U, V be connected open subsets in $\mathbb{R}^n$, and let $\psi : V \to U$ be a diffeomorphism. Suppose that ω is a differential n-form on U with compact support. Then*

$$\int_V \psi^* \omega = \epsilon(\psi) \int_U \omega$$

where $\epsilon(\psi)$ is $+1$ if ψ is orientation-preserving and -1 if ψ is orientation-reversing.

Proof. Denote by $x_1, \ldots, x_n$ the Euclidean coordinates on U and by $y_1, \ldots, y_n$ those on V. Let $J(\psi)$ be the Jacobian determinant of ψ. Write $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$. We have

$$\int_V \psi^* \omega = \int_V f \circ \psi(y) J(\psi)(y) dy_1 \wedge \cdots \wedge dy_n.$$

On the other hand, by the change of variables formula for integral, we have

$$\int_U \omega = \int_U f(x) dx_1 \cdots dx_n = \int_V (f \circ \psi)(y) |J(\psi)(y)| dy_1 \cdots dy_n.$$

Clearly, these two formulas differ by the multiplicative factor $\epsilon(\psi)$. $\square$

Now suppose that M is an oriented n-manifold and that ω is a differential n-form on M which is supported in a coordinate patch of M. We choose such a patch $V \subset M$ with $\psi : V \cong U \subset \mathbb{R}^n$ compatible with the orientation of M. We define $\int_M \omega$ to be $\int_U (\psi^{-1})^* \omega$. By the previous lemma, this is independent of the choice of oriented coordinate patch U, but it does depend on the orientation of M – the integral changes sign if we reverse the orientation of M.

More generally, let ω be a compactly supported differential n-form on an oriented n-manifold M. There is a partition of unity $\{\lambda_U\}$ on M subordinate to the open covering of M by coordinate patches. We can write

$$\omega = \sum_U \lambda_U \omega$$

where $\lambda_U\omega$ has support in the coordinate patch U. Since ω is compacty supported, we can arrange it so that this is a finite sum. According to the previous discussion, $\int_M \lambda_U\omega$ is defined for each U and hence we can define

$$\int_M \omega = \sum_U \int_M \lambda_U\omega.$$

Exercise H.0.6. *Show the above definition is independent of the covering and the partition of unity. Show that $\int_M \omega$ changes sign when we reverse the orientation of M.*

Thus, if M is a compact oriented k-manifold and ω is a differential k-form, then $\int_M \omega$ is defined.

We need a generalization of this to smooth manifolds with boundary. Suppose that M is an oriented smooth k-manifold with boundary and that ω is a k-form with compact support. Then we have $\int_M \omega$ defined as before: we take a partition of unity subordinate to a covering by coordinate charts so that it suffices to compute the integral of a form supported in a single coordinate chart. Even if the chart is an open subset of half-space, the integral is still defined.

We define the induced orientation of ∂M as follows. At each point $x \in \partial M$ we consider an orientation for TM_x given by an ordered basis for this vector space whose first vector points out of M and all the others are tangent to the boundary. The restriction to the subset consisting of all but the first vector then gives the induced orientation of ∂M at x.

Theorem H.0.7. (Stokes' Theorem) *Let M be a compact oriented k-manifold possibly with boundary. Give ∂M the induced orientation. Let ω be a differential $(k-1)$-form on M. Then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

Proof. Cover M by finitely many coordinate charts which are cubes in $\mathbb{R}^n$ and take a partition of unity λ_i subordinate to this covering. Then $\omega = \sum \omega_i$ where $\omega_i = \lambda_i\omega$ is supported in the i^{th} coordinate patch. Clearly, establishing the result for each ω_i will establish it for ω since both sides are additive. Thus, it suffices to prove the result for a $(k-1)$-form ω in a cube I^k in $\mathbb{R}^k$. We write

$$\omega = \sum_i f_i(x_1, \ldots, x_k)dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

Again by linearity it suffices to consider the terms one at a time, so we may suppose that

$$\omega = f(x_1, \ldots, x_k)dx_1 \wedge \cdots \wedge dx_{k-1}.$$

Then the integral of ω along all the faces of the boundary except the ones where $x_k = 0$ and $x_k = 1$ vanish. The boundary orientation on the face where $x_k = 1$ agrees with $(-1)^{k-1}$

times the orientation induced from $\mathbb{R}^{k-1}$ whereas the boundary orientation on the face $x_k = 0$ is the opposite one. Thus, these integrals add up to:

$$(-1)^{k-1}\int_{I^{k-1}} f(x_1,\ldots,x_{k-1},1) - f(x_1,\ldots,x_{k-1},0)dx_1\ldots dx_{k-1}.$$

On the other hand,

$$d\omega = \frac{\partial f}{\partial x_k}dx_k \wedge dx_1 \wedge \cdots \wedge dx_{k-1} = (-1)^{k-1}\frac{\partial f}{\partial x_k}dx_1\cdots dx_k,$$

and by Fubini's theorem its integral is equal to

$$(-1)^k \int_{I^{k-1}} \left(\int_0^1 \frac{\partial f}{\partial x_k}\right) dx_1\cdots dx_{k-1}.$$

By the Fundamental Theorem of Calculus these expressions are equal. □

Suppose that N is a smooth manifold and that ω is a differential k-form on N. Let $f : M \to N$ be a smooth map of a compact, oriented k-manifold (possibly with boundary) into N. We define $\int_M \omega$ to be the integral of $f^*\omega$ over M.

Exercise H.0.8. *Show that this operation is linear in ω and show that if η is a $k-1$-form, then*

$$\int_{\partial M}\eta = \int_M d\eta.$$

This is a more general form of Stokes' theorem.

Exercise H.0.9. *Define smooth n-manifolds with corners as being modeled on open subsets in Q^n, the positive quadrant in $\mathbb{R}^n$. Extend Stokes' theorem to manifolds with corners mapping smoothly into M.*

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